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Information Dissemination in Power-constrained Wireless Networks

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Technical Report Number UH-CS-05-16

July 21, 2005

Keywords: wireless networks, information dissemination, broadcast capacity, information diffusion, percolation theory

Abstract

Dissemination of common information through broadcasting is an integral part of wireless network operations such as query of interested events, resource discovery and code update. In this paper, we characterize the behavior of information dissemination in power-constrained wireless networks by defining two quantities, i.e., *broadcast capacity* and *information diffusion rate* and derive fundamental limits in both random extended and dense networks. We find that using multihop relay, the rate of broadcasting continuous stream is $\Theta((log(n)^{-\frac{\alpha}{2}})$ in extended networks; while direct single-hop broadcast is efficient for dense networks. Furthermore, regardless of the density, information can diffuse at constant speed, i.e., $\Theta(1)$ in both extended and dense networks. The theoretical bounds obtained and proof techniques are instrumental to the modeling and design of efficient wireless network protocols.

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Index Terms

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I. Introduction

A wireless ad hoc network consists of nodes that communicate with each other over a shared wireless channel. In recent years, there has been growing interest in understanding the fundamental capacity limits of such networks [4], [7]–[9], [16], [20], [24], [25]. Capacity results are not only of interest from a theoretical perspective but also provide guidelines for protocol design in wireless ad hoc networks [11]. In their seminal work, Gupta and Kumar [8] show that assuming each node can transmit with a constant rate, the per-node throughput capacity of a random wireless ad hoc network with n static nodes decreased with n as $O(\frac{1}{\sqrt{n}})$. Since then, many work [4], [7], [16], [20], [24], [25] has been done towards deriving capacity bounds of wireless networks under different modalities of communication and/or assumptions, such as with mobility, using infinite wireless bandwidth etc.

While much has been done to derive the throughput capacity of unicast communication among randomly selected node pairs, there have been little efforts in understanding how fast *common information* can be disseminated throughout the network via multihop relays. In infrastructureless networks such as wireless sensor networks and wireless mesh networks, broadcasting is a common form of operation. Examples are code update [10], query of interested events, network maintenance and resource discovery. The speed and rate of information disseminating are important aspects of protocol behavior in wireless networks. For example, in on-demand routing protocols such as DSR or AODV, the timeout value of out-standing route requests should be selected to reflect the maximum amount of time a route discovery message takes to be delivered through the network. Otherwise, the source may timeout prematurely and inject unnecessarily additional traffic in the network. Consider a digital video broadcast service operating in a community wireless mesh network, the maximum rate the source can transmit the video data determines the quality of reception at each subscriber.

In this paper, we study the fundamental theoretical limits of information dissemination in power-constrained wireless networks under a generalized physical model. To characterize the information dissemination process, we rigorously define two quantities, i.e., *broadcast capacity* C_b and *information diffusion rate* ω_b. They represent two correlated and yet distinctive aspects of information dissemination processes. We derive asymptotic properties of the two qualities in both extended and dense networks (formally defined in Section II) and compare the results to a direct broadcast scheme. We find that using multihop relay, the rate of broadcasting continuous stream is $\Theta((log(n)^{-\frac{\alpha}{2}})$ in extended networks; while direct single-hop broadcast is efficient for dense networks. Furthermore, regardless of the density, information can diffuse at constant speed, i.e., $\Theta(1)$ in both extended and dense networks. The proofs to obtain the lower bounds provide insights on the design of efficient protocols to disseminate information in multihop wireless networks.

The rest of the paper is organized as follows. In Section II, we present a review of existing results in the capacity limits of multihop wireless networks. We formally define the information dissemination problem and summarize the main results in Section III. The bounds for *broadcast capacity* C_b and *information diffusion rate* ω_b are derived for both extended and dense networks in Section IV and Section V, respectively. Finally, we conclude the paper in Section VI with a list of future work.

II. Related Work

There has been vast literature on the capacity limits of multihop wireless networks. A thorough survey can be found in [25] as of the time the referred paper was published. In this section, we first summarize the most recent results on capacity limits and then review a few work that most pertain to the results and methodology in this paper.

Capacity bounds in wireless networks are ‘primarily concerned with the scaling laws of how fast information can be transported over distance with respect to the number of communicating nodes. Typically, there are two ways of letting the number of nodes n tend to infinity [4]. One can either keep the area on which the network is deployed constant, and make the node density λ tend to infinity (termed *dense network*); or one can keep the node density λ constant, and increase the area to infinity (termed *extended networks*). For both of these settings, *information theoretic* upper bounds are obtained by allowing arbitrary (physical layer) cooperative relay strategies; while *network theoretic* bounds are usually derived through constructive methods by assuming certain relationship between link capacity and interference models, and applying graph theoretical results on the resulting weighted graphs.

Information theoretical bounds for unicast communication: For extended networks, Xie and Kumar [24] prove $\Theta(\frac{1}{\sqrt{n}})$ bit/sec per node, for arbitrarily located nodes satisfying a minimum distance constraint, and a power attenuation function that has a power law exponent $\alpha > 6$ or an exponential absorption. When $\alpha < 2$, the information-theoretical throughput capacity is unbounded. When nodes are randomly located, Leveque and Preissmann [13] show an upper bound of $O(\frac{1}{n^{1/2-1/\alpha}})$ for $\alpha > 2$.

Network theoretical bounds for unicast communication: A constructive strategy is proposed for extended networks of randomly located nodes which achieves a per-node capacity of $\Omega(\frac{1}{\sqrt{n}})$, when $\alpha > 2$. This is an improvement over earlier results of $\Omega(\frac{1}{\sqrt{nlog(n)}})$ [8]. The main idea of the strategy is of have a wireless backbone of nodes that carry packets across the network at constant rate, using short hops, and to drain the rest of the traffic to the wireless backbone using single hops of longer length. The existence of the backbone is due to percolation theory [6]. In dense networks, the same strategy is still applicable achieving a per-node capacity of $\Omega(\frac{1}{\sqrt{n}})$. Since the lower bound is on the same order as the information theoretic upper bound in extended networks for $\alpha > 6$, multihop relay without physical layer cooperative strategies is order optimal [11]. It is further proved in [1], in dense networks, the per-node throughput capacity is $O(n)$ bit·meter/s for arbitrarily chosen sender-receiver pairs under a generalized physical model.

The work in [20], [25] considers the capacity bound in ultra wide band channels. When the bandwidth grows an order of magnitude faster than n, interferences from concurrent transmitters are negligible compared to the background noise. It has been proved in [25], that each node can achieve a throughput capacity of $\Theta(n^{(\alpha-1)/2})$ in dense networks.

Network theoretical bounds for many-to-one communication: Giridhar and Kumar [5] study the problem of how nodes in the network should cooperate to efficiently compute a desired function $f(x_1, x_2, \ldots, x_n)$, and to make it available at a specified sink node. It is proved that the maximum rate of downloading the frequency histogram in a random planar multi-hop network with n nodes is $O(\frac{1}{log(n)})$. Type-sensitive functions such as mean and median can be computed at a rate $O(\frac{1}{n})$. Type-threshold functions such as max, min and range can be computed at a rate $O(\frac{1}{log(n)})$. The above results are obtained using the protocol model proposed in [8], where concurrent transmissions in the vicinity of the receiver collide.

Computing and communicating functions over sensor networks are the inverse problem of what we consider in the paper. While we adopt some of the proof techniques in [5], our problems require different treatments.

Information theoretical results for one-to-many communication: In [12], Khishi *et al* derive the capacity theorem for cooperative multicasting in large wireless networks with slowly fading channel. They assume a different channel model, i.e., the channel gain between node $i \in \{0, 1, ..., n\}$ and node $j \in \{1, 2, \ldots, n\}$ is denoted by h_{ij} which is assumed to be $\mathcal{CN}(0, 1)$, independent of all other gains and constant throughout. Furthermore, it is assumed that the set of transmitting nodes are subject to a sum (as opposed to individual) power constrain of P. It is proved in [12], that the rate R achievable for the cooperative multicasting system is $log(1 + \frac{P}{N_0})$ as $n \to \infty$. The key idea behind the cooperative diversity based scheme is a two-phase process. First, the source s broadcast to a β fraction of "good" receivers at infinity rate if β is chosen properly. Second, the set of "good" receivers perform beam forming toward the rest of the receivers.

The above results do not consider the effect of signal attenuation over distance. Furthermore, the assumption of independent channel gains among all node pairs is no longer valid as the number of nodes goes to infinity. The network information bounds for one-to-many communication remain to be an open problem under more realistic channel models.

Efficient one-to-many communication protocols: One of the most widely used strategy to disseminate data in the network is through flooding. A host, on receiving a broadcast message for the first time, rebroadcasts the message to its neighbors. As articulated in [17], flooding can cause broadcast storm, i.e., redundant rebroadcasts, channel contention and collisions in the network. Several mechanisms have been proposed to alleviate the broadcast storm problem. One solution is to jitter the transmission of rebroadcasts to avoid collisions [17]. In addition, redundant transmissions can be suppressed based on the number of broadcast messages received, relative location of neighbors etc. For example, in [10] a counter based scheme is proposed. A node refrains from re-broadcasting upon overhearing multiple broadcasts from its neighbors. In [15], nodes rebroadcast message with a probability p as a function of the hop count from the source. It is shown the percentage of nodes receive the broadcast messages follows a bimodal distribution.

The work in [3], [14], [21]–[23] constructs energy-efficient multicast and broadcast trees by selectively pruning the network connectivity graph. The objective is to minimize the total energy consumption in broadcasting/multicast packets in multihop wireless networks. In comparison, in this paper, we focus on deriving the maximum rate and speed of information dissemination. The resulting dissemination tree may not be energy optimal.

Our work is in part motivated by the open problem raised in *Deluge* [10]. *Deluge* is a reliable data dissemination protocol for propagating large data objects from one or more source nodes to many other nodes over a multihop, wireless sensor network. *Deluge* builds on a density-aware (using counter schemes), epidemic propagation mechanisms. The authors state in [10] that, dissemination (though multihop relay) is inherently slower than single path propagation and identify establishing a tight lower bound as an open problem. In this paper, we solve the open problem by deriving both the upper and lower bound of the speed of dissemination through rigorous analysis. The proof to obtain the lower bound can be potentially used for efficient data dissemination in multihop wireless networks.

III. Problem Statement and Main Results

In this section, we formally define the problems of information dissemination in power-constrained wireless networks and summarize the key results.

A. Assumptions

We construct a random extended network by placing nodes according to a Poisson point process $\mathcal{P}$ of unit rate on a 2-D plane. Let $B(n)$ denote the box $[0, \sqrt{n}]^2$, and the set $\mathcal{P}_n := \mathcal{P} \cap B(n)$, a Poisson point process of unit rate on $B(n)$. Similarly, we construct a dense network by placing nodes according to a Poisson point process of rate n and set $B(n) = [0, 1]^2$. The following assumptions are made to facilitate analysis:

1) Nodes are individually power-constrained. Denote by $P_i \geq 0$ the power used by node i. Thus, $P_i \leq P_{max}$.
2) The channel follows ambient Gaussian noise model with power spectral density of $N_0/2$ and the signal attenuation of $\rho^{-\alpha}$, where ρ is the distance between the transmitter and receiver(s), and α is the path loss exponent.

3) When common information is directly broadcast from a node i to a set of receivers $\mathcal{R}$, capacity-achieving Gaussian channel codes are assumed to support the achievable rate of the worst receiver, i.e.,

$$r_i = \min_{j \in \mathcal{R}} Blog(1 + SINR_{ij}), \tag{1}$$

where $SINR_{ij} = \frac{P_i \rho_{ij}^{-\alpha}}{BN_0 + \sum_{k \in I} P_k \rho_{kj}^{-\alpha}}$, I is the set of nodes that are simultaneously transmitting and B is the bandwidth of the channel.

4) No cooperative relay strategy is employed at the physical layer.

The third assumption defines the rate for single-hop broadcast communication. It is in analogy to link capacity in point-to-point communication. The sender determines which set of receivers it needs to reach and chooses a coding scheme and transmission power level accordingly such that the node with the least SINR can successfully decode the message. In general, the isobars of SINR associated with a single transmitter can be of very complex shape. Only in the special case when there is no interference and a free-space propagation model is assumed, the isobars of SINR are circles centered at the transmitter. Thus, each broadcast transmission of rate r_i can be associated with a radius t_i, such that nodes within the range t_i of the transmitter i all have $SNR \geq e^{\frac{r_i}{B}} - 1$.

B. Problem statement

Consider a source node s that needs to disseminate data to all other nodes, we are primarily interested in the following two quantities:

*Definition 1 (***Broadcast capacity (C_b)***):* C_b is the maximum rate (bit/s) that s can *continuously* broadcast information to all other nodes in $B(n)$ such that all nodes can correctly receive the information in finite time.

*Definition 2 (***Information Diffusion Rate (ω_b)***):* For a single message of L bit, let $T(n)$ be the time that the message reaches all nodes in $B(n)$. Let $|B(n)|$ be the area of $B(n)$ (e.g., in extended and dense networks, $|B(n)|$ are n and 1 respectively). Information diffusion speed of a single message is defined as $\omega_b = \sup\{\frac{L \cdot |B(n)|^{\frac{1}{2}}}{T(n)}\}$ ($bit \cdot meter/s$), the supremum is taken over all possible routing schemes and transmission schedules.

The two quantities are correlated and yet characterize different aspects of the information dissemination process. C_b reflects how fast *continuous stream* of data can be delivered to all nodes. Examples of continuous data stream are software upgrade and video broadcasting. The constraint is that none of the intermediate nodes is overloaded. At any time of the process, nodes may be transmitting different segments of the data (in packets). We will comment on the buffer requirement in later section. Information diffusion rate ω_b, on the other hand is determined by two factors, 1) how fast and how far a message can be transported in one transmission and 2) how many relays are required to reach the furthest nodes. It models after resource discovery, query propagation operations in wireless networks. When nodes are unevenly distributed in space, information may diffuse at heterogeneous speeds along different directions at various locations. The above definition considers the time to reach all nodes in $B(n)$.

Without loss of generality, we assume the source is at the origin in the 2-D plane.

C. Main results

We adopt the asymptotic notations in complexity theory, i.e., $O(\cdot)$ describes asymptotic upper bounds, $\Omega(\cdot)$ for asymptotic lower bounds, and $\Theta(\cdot)$ for asymptotically tight bounds. The key results in this paper are summarized as follows.

Theorem 1: Assume a power attenuation function of exponent $\alpha > 2$. In extended networks, the broadcast capacity

$$C_b = P_{max} \Theta(log(n)^{-\frac{\alpha}{2}}), \tag{2}$$

In dense networks,

$$C_b = O(log(n) + log(P_{max})) \text{ and } C_b = log(P_{max})\Omega(1). \tag{3}$$

Theorem 2: Assume a power attenuation function of exponent $\alpha > 2$. In both extended and dense network, the information diffusion rate is constant, i.e.,

$$\omega_b = \Theta(1). \tag{4}$$

To see the significance of the above results, we consider a base line where the source node transmits messages to all other nodes through direct broadcasts.

In an extended network, let node $z \in B(n) = [0, \sqrt{n}]^2$ be the furthest node from the source node s. Then, we have

Lemma 1: $||z|| \geq \sqrt{n}$ with high probability, where $||\cdot||$ is the Euclidean norm and $||z||$ is the distance between z and s.

Proof: Let A be the region in $B(n)$ outside the disk centered at the s with radius $\sqrt{n}$. The area of A is given by $|A| = (1-\pi/4)n$. By the definition of Poisson point process, $P\{||z|| \leq \sqrt{n}\} = P\{$There is no node in area A$\} = e^{-|A|} = e^{-(1-\pi/4)n}$. As $n \to \infty$, $P\{||z|| \leq \sqrt{n}\} = 0$.

■

Therefore, the rate that node z can receive the message is upper bounded by $r_z \leq Blog(1+\frac{P_{max}n^{-\frac{\alpha}{2}}}{BN_0})$ using direct broadcast. We have $C_b \leq Blog(1+\frac{P_{max}n^{-\frac{\alpha}{2}}}{BN_0})$ and $\omega_b \leq B\sqrt{n}log(1+\frac{P_{max}n^{-\frac{\alpha}{2}}}{BN_0})/L$. Clearly, $C_b = P_{max}\Theta(n^{-\alpha})$ while $\omega_b = P_{max}\Theta(n^{\frac{1-\alpha}{2}})$ as $n \to \infty$. By the same argument, in dense networks, $C_b \leq Blog(1+\frac{P_{max}}{BN_0}) = P_{max}\Theta(1)$ and $\omega_b = \frac{L\cdot 1}{L/C_b} = P_{max}\Theta(1)$.

Compared with the results in Theorem 1 and 2, direct broadcasts are suboptimal in disseminating information in extended wireless networks. It is beneficial to use multi-hop relays. However, in dense networks, the best strategy to propagate information appears to be through direct broadcasts. In the next two sections, we focus on deriving the bounds for extended networks, the results for dense networks are included for completeness.

IV. Bounds on Broadcast Capacity

A. Upper bound

In broadcasting data in a multihop network, messages are forwarded on a tree that spans the network rooted at the source. While the broadcast tree can change and a node may have different on-tree parent nodes over time, a node cannot receive messages at a rate faster than the capacity of its fastest link. A nearest neighbor graph $NNG(n)$ is defined as a acyclic graph that links each vertex in $\mathcal{P}_n$ to its nearest neighbor in Euclidean distance. Let $\mathcal{N}_1(u)$ be u's neighbor on $NNG(n)$. Let $C(u) = Blog(1+\frac{P_{max}\rho_{u\mathcal{N}_1(u)}^{-\alpha}}{BN_0})$, i.e., the link capacity to the nearest neighbor. By the above argument, $C_b \leq \min_{u\in\mathcal{P}} C(u)$.

It is proven [18] that given a Poisson process of rate n on a 2-D unit area, the longest edge of NNG, M_n satisfies $\lim_{n\to\infty} P[n\pi M_n^2 - log(n) \leq \gamma] = exp(-e^{-\gamma})$. By scaling the rate of Poisson point process to 1 and the area to n, it is easy to show that

$$\lim_{n\to\infty} P[\pi \cdot M_n^2 - log(n) \leq \gamma] = exp(-e^{-\gamma}). \tag{5}$$

As long as $\gamma \to -\infty$ as $n \to \infty$, the right hand side approaches 0. Let $\gamma = -log(n)/2$ and define $\epsilon_0 \triangleq \frac{1}{2\pi}$. We have

$$\lim_{n\to\infty} P[M_n > \sqrt{\epsilon_0 log(n)}] = 1. \tag{6}$$

Consider one node u which shares the longest edge on $NNG(n)$. The data rate that node u can receive is bounded by,

$$C(u) = Blog_2(1+\frac{\frac{P_{max}}{M_n^{\alpha}}}{BN_0}) < Blog_2(1+\frac{P_{max}(\epsilon_0 \cdot log(n))^{-\frac{\alpha}{2}}}{BN_0}) \tag{7}$$

Therefore, $C_b = \min_{v\in\mathcal{P}} P_{max}C(v) = O(log(n)^{-\frac{\alpha}{2}})$ as $n \to \infty$.

As argued later in Section V, the upper bound is not tight. However, we cannot be conclusive regarding the optimality of the broadcast capacity results using direct broadcasts in dense networks.

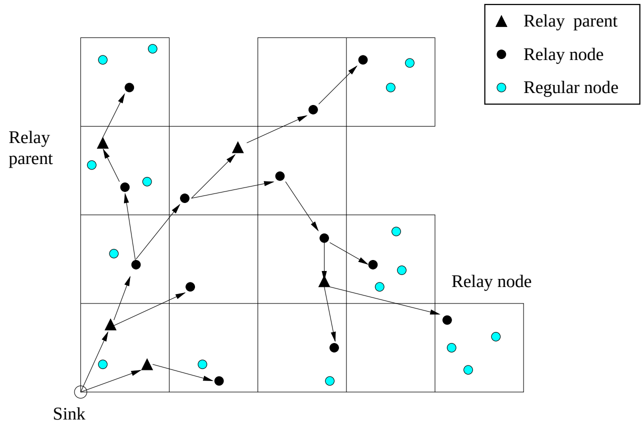

Fig. 1. A spanning tree on cell graph with relay nodes, relay parents. The solid round dots and triangle dots are relay nodes and relay parents respectively. The light dots are regular nodes, which do not participate in the message forwarding process. Note that a relay node and relay parent may co-exist on the same node.

B. Lower Bound

Let $\gamma = log(n)$ and define $\epsilon_1 \triangleq \frac{2}{\pi}$. From Eq. (5), it is easy to show that

$$\lim_{n\to\infty} P[M_n \leq \sqrt{\epsilon_1 log(n)}] = 1. \tag{8}$$

Let $r = \sqrt{\epsilon_1 log(n)}$. Define a random geometric graph $G(n,r)$ on $B(n)$ where a pair of nodes are connected by an edge if their Euclidean distance is less than r. Since the longest edge on NNG is asymptotically of the same length as that on a minimal spanning tree [18] in a 2-D plane, $G(n,r)$ is connected.

Consider a lattice (cell) of edge length $l = r/\sqrt{2}$. A cell graph G_c consists of the set of non-empty cells as vertices, with two cells being adjacent if there exist two nodes within each cell respectively, which share an edge in $G(n,r)$. The following result characterizes the relationship between the cell graph and the original graph. Define a many-to-one mapping $\mathfrak{c}(x) : \mathbb{R}^2 \to \mathbb{Z}^2$ as $(\lfloor \frac{x_1}{l} \rfloor, \lfloor \frac{x_2}{l} \rfloor)$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

Lemma 2: The cell graph G_c is connected iff $G(n,r)$ is connected. Furthermore, let $l_G(x \leftrightarrow y)$ be the length of a path joining two connected vertices x and y on $G(n,r)$, and $l_{G_c}(\mathfrak{c}(x) \leftrightarrow \mathfrak{c}(y))$ be the number of edges of the corresponding path joining $\mathfrak{c}(x)$ and $\mathfrak{c}(y)$ on the cell graph. Then,

$$\lfloor \frac{l_G(x \leftrightarrow y)}{3} \rfloor \leq l_{G_c}(c(x) \leftrightarrow c(y)) \leq l_G(x \leftrightarrow y) \tag{9}$$

Proof: Let $(x = x(1), x(2), \ldots, x(l_G(x \leftrightarrow y)) = y)$ be a path in $G(n,r)$. Clearly, $(\mathfrak{c}(x), \mathfrak{c}(x(2)), \ldots, \mathfrak{c}(y))$ is also connected (with the removal of duplicated vertices) in the cell graph. Now consider two nodes x and y in $G(n,r)$, their corresponding cells $\mathfrak{c}(x)$ and $\mathfrak{c}(y)$ are connected by a path $(c_1 = \mathfrak{c}(x), c_2, \ldots, c_{l_{G_c}(\mathfrak{c}(x)\leftrightarrow\mathfrak{c}(y))}) = \mathfrak{c}(y))$. By the definition of the cell graph, there exist two nodes u and v such that $\mathfrak{c}(u) = c_1$, $\mathfrak{c}(v) = c_2$ and $||u - v|| \leq r$. Since $||x - u|| \leq r$, we have x is connected to v. Repeating this procedure for each cell, we can construct a path in $G(n,r)$ from x and y.

To prove the second part of the lemma, let us consider an edge e on the cell graph G_c. Its inverse image in $G(n,r)$ is a path of length no greater than 3 and no less than 1. By summing all the edges on the path in the cell graph, we get the result in Eq. (9).

■

Following the notation in [5], we designate the cell with the source node as the root, and construct a spanning tree on the cell graph. In each cell, let a node u in cell c, which is adjacent to a node v in its parent cell (in the spanning tree) be the relay node of c, and designate v as the corresponding relay parent of u in the parent cell. Clearly, there is one relay node per cell and at most $5^2 - 1 = 24$ relay parents (corresponding to each non-empty neighboring cell). An illustrative example is given in Figure 1.

Define the following transmission schedule in a slotted system. In every slot (to be determined next),

1) First, a relay node u in cell c broadcasts the latest message received from its relay parent to all nodes in cell c.

2) Second, each relay parent in c unicasts the message to its child cells sequentially.

The length of each slot needs to accommodate at most 25 message transmissions. What remains to be determined is the rate of each transmission.

Lemma 3: For any given integer $k \geq 0$, under the generalized physical layer model in Section III-A, there exists a TDMA scheduling, such that one node per square of edge length l can transmit to nodes located within a radius of k squares (in Manhattan distance) with fixed rate $R(k)$ given as follows.

$$R(k) \geq \frac{B}{(2+2k)^2} log(1 + \frac{P_{max}}{BN_0(l(k+1))^\alpha + K' P_{max}}), \tag{10}$$

where K' is a constant independent of k and l. The result holds for $\alpha \geq 2$.

Proof: The result is in essence an extension of Theorem 4 and Theorem 5 in [4] to the broadcast scenario. For completeness, we include the proof in Appendix I.

■

Theorem 3: When $\alpha \geq 2$, as $n \to \infty$, $C_b = P_{max}\Omega(log(n)^{-\frac{\alpha}{2}})$ is achievable.

Proof: Let $k = 2$ and replace $l = \sqrt{\epsilon_1 log(n)/2}$ (recall $\epsilon_1 \triangleq \frac{2}{\pi}$) in Eq. (10). The achievable rate of each transmission is $P_{max}\Omega((\epsilon_1 log(n))^{-\frac{\alpha}{2}}) = P_{max}\Omega(log(n)^{-\frac{\alpha}{2}})$ as $n \to \infty$. We need to divide this throughput by the number of transmissions in each slot. Since there are at most 25 transmissions in each slot, $C_b = P_{max}\Omega(log(n)^{-\frac{\alpha}{2}})$.

■

Combining the upper and lower bound, we have Theorem 1, i.e., $C_b = P_{max}\Theta(log(n)^{-\frac{\alpha}{2}})$, for $\alpha \geq 2$.

Buffer requirement: Since each relay node or relay parent only has one upstream node on the dissemination tree and each on-tree node transmits at the same rate, a buffer size of one is sufficient for a node to hold a packet till its own transmission slot.

C. Bounds for Dense Networks

In dense networks where the distribution of nodes follow a Poisson process on rate n, the length of the longest edge in NNG and the minimal spanning tree satisfies $M_n \leq \sqrt{\frac{\epsilon_0 log(n)}{n}}$. Therefore, the broadcast capacity $C_b \leq Blog(1 + \frac{P_{max}M_n^{-\alpha}}{BN_0}) = O(log(P_{max}(\frac{n}{\epsilon_0 log(n)})^{\frac{\alpha}{2}})) = O(log(n))$, as $n \to \infty$.

Theorem 4: The broadcast capacity is $O(log(n))$.

This upper bound is loose because it does not consider the interference from other transmitters. We conjecture that the upper bound of the broadcast capacity is $O(1)$ in dense networks and leave it as a future work.

To get the lower bound, replace $l = \sqrt{\frac{\epsilon_1 log(n)}{n}}$ and $k = 2$ in Lemma 3, and we have $C_b = \Omega(log(P_{max}))$ for $\alpha \geq 2$ as $n \to \infty$.

V. BOUNDS ON INFORMATION DIFFUSION RATE

Recall in Section III-B, information diffusion rate ω_b is defined as the inverse of the time it takes for a single bit to travel one meter, i.e., $\omega_b = \sup\{\frac{L \cdot \sqrt{|B(n)|}}{T(n)}\}$. It characterizes how fast information diffuses throughout the network.

A. Upper Bound

To derive the upper bound, let us first consider the progression of a single transmission. To transmit a message of length L from the source node s to point $y \in \mathbb{R}^2$ using transmission power P_{max}, the transmission time is at least $t_{1hop} = L/log(1 + \frac{P_{max}||y||^{-\alpha}}{BN_0})$. Therefore, the "velocity" of message diffusion v is,

$$v = \frac{L||y||}{t_{1hop}} = ||y||log(1 + \frac{P_{max}||y||^{-\alpha}}{BN_0}).$$

Define

$$v_{max} \triangleq \max_y \frac{L||y||}{t_{1hop}} = \max_y ||y||log(1 + \frac{P_{max}||y||^{-\alpha}}{BN_0}).$$

v_{max} can be obtained using simple optimization techniques and is independent of n.

Theorem 5: The information diffusion rate $\omega_b \triangleq sup\{\frac{L\cdot\sqrt{n}}{T(n)}\}$ is upper bounded by a constant, i.e., $\omega_b = O(1)$.

Proof: Consider a node z that is the furthest node from s. The minimum time for z to receive the message is $L||z||/v_{max} \geq \frac{\sqrt{n}}{v_{max}}$ *w.h.p* (Lemma 1). Therefore, $T(n) \geq \frac{\sqrt{n}}{v_{max}}$. Consequently, $sup\{\frac{L\cdot n}{T(n)}\} \leq v_{max}$. Since v_{max} is a constant independent of n, the result holds. ■

B. Lower Bound

Next, we derive the lower bound using a constructive proof. The key idea is to first flood the message on a connected "fast" backbone of nodes that span $B(n)$; then, selected nodes on the backbone broadcast the message to its neighbors concurrently. The "fast" backbone can deliver message to the furthest node (through multihop relay) in order $O(\sqrt{n})$ time; and the single-hop broadcast takes logarithmic time $O(log(n))$. Let $\mathcal{X}(n)$ be the "fast" backbone and $G(\mathcal{X}(n), r)$ is a subgraph of $G(n, r)$ with vertices in $\mathcal{X}(n)$ and edges between node pairs within distance r.

Let $E(n)$ be the event that there exists a connected "fast" backbone $\mathcal{X}(n)$ that satisfies the following properties:

P1 There exists a constant ϵ_2 such that $G(\mathcal{X}(n), \epsilon_2)$ is connected as $n \to \infty$.
P2 $\forall x \in \{\mathcal{P}_n \backslash \mathcal{X}(n)\}$, $\exists y \in \mathcal{X}(n)$, s.t., $||x - y|| \leq \epsilon_2 log(n)$.
P3 $\forall x, y \in G(\mathcal{X}(n), \epsilon_2)$, there exists a self-avoiding path of length (in hop count) $O(||x - y||)$.

The main result in this section is summarized as follows:

Theorem 6: As $n \to \infty$, $P\{E(n)\} = 1$.

Consider one such $\mathcal{X}(n)$. Designate a node $s' \in \mathcal{X}(n)$ as the proxy for s, which satisfies $||s - s'|| \leq \epsilon_2 log(n)$. Property (P2) guarantees the existence of such a node. If $s \in \mathcal{X}(n)$, then $s' = s$. We partition the 2-D plane into lattice L_1 with edge length $\epsilon_2/\sqrt{2}$. Construct a cell graph $G_c(\mathcal{X}, \epsilon_2)$ on $G(\mathcal{X}, \epsilon_2)$, a shortest-path spanning tree rooted at s' and designate relay nodes and relay parents in the same fashion as in Section IV.

Furthermore, we construct another lattice L_2 of edge length $\frac{\epsilon_2 log(n)}{\sqrt{2}}$. For each cell c' in L_2 that contains at least one node in $\mathcal{X}(n)$, select a node $u \in c' \cap \mathcal{X}(n)$ as the *relay seed*.

Next we describe the procedure to disseminate information using the "fast" backbone. Starting from the source node,

Stage I: The source node s unicasts the message to s'. (Note that if $s = s'$, then this stage is skipped).

Stage II:

1) A relay node u in cell $c \in G_c(\mathcal{X}, \epsilon_2)$ broadcasts the message received from its relay parent to nodes in cell c.
2) Each relay parent in c unicasts the message to its child cell sequentially.

Stage III:

1) Each relay seed broadcasts the message to all nodes in radius $2\epsilon_2 log(n)$.

The following result states the correctness of the protocol.

Lemma 4: By the end of Stage III, all nodes have received the message.

Proof: By Property (P1), $G_c(\mathcal{X}, \epsilon_2)$ is connected. Thus, at the end of Stage II, the message has been disseminated throughout nodes in $\mathcal{X}$. Consider a node $x \in \{\mathcal{P}_n \backslash \mathcal{X}(n)\}$. There exists a node $y \in \mathcal{X}(n)$ such that $||x - y|| \leq \epsilon_2 log(n)$ due to Property (P2). If y is a relay seed, x will receive the message from y in Stage III. Otherwise, there must exist a relay seed, say, $z \in \mathcal{X}(n)$ in the same cell as y in L_2. Since $||z - y|| \leq \epsilon_2 log(n)$, $||x - z|| \leq 2\epsilon_2 log(n)$ by triangle inequality. Therefore, x will receive the message from x in Stage III. ■

An example of Stage II operation is given in Figure 2. At the end of Stage I, node A and B on the "fast" backbone have received the message. Node B is serving as the *relay seed* and broadcasts the message in its $2\epsilon_2 log(n)$ neighborhood.

Let the time spent in Stage I, II and III be t_1, t_2 and t_3 respectively.

Lemma 5: $\forall x \in \mathcal{X}(n)$, the time t_2 it takes to reach x from s' in Stage II is $O(\sqrt{n})$ as $n \to \infty$.

Proof: From Property (P3), $\forall x \in G(\mathcal{X}(n), \epsilon_2)$, there exists a self-avoid path of length $O(||x - s'||)$ that joins x and s' on $G(\mathcal{X}(n), \epsilon_2)$. The length of the corresponding path on the cell graph $G_c(\mathcal{X}(n), \epsilon_2)$ is at most

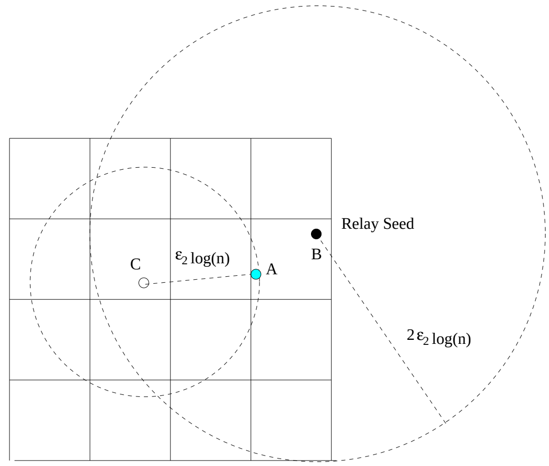

Fig. 2. At the end of Stage II, node A and B have received the messages. Relay seed B then broadcast the message in its neighborhood within radius $2\epsilon_2 log(n)$.

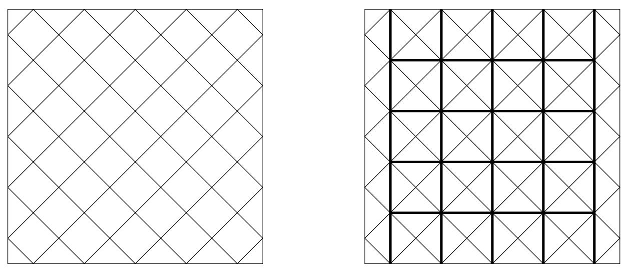

Fig. 3. Construction of the bond percolation model. Each cell on the left-hand side is open, if there is at least a Poisson point inside it, closed otherwise. This corresponds to associate an edge to each cell, traversing it diagonally, as depicted on the right-hand side, and let the edge be either open or closed according to the state of its corresponding cell.

$3O(||x-s'||)$ due to Lemma 2. Since $\max_{x \in G(\mathcal{X}(n), \epsilon_2))} ||x-s'|| \leq \sqrt{n}$, the maximum hop distance from s' to x is bounded by $O(\sqrt{n})$.

To determine the rate of each transmission, let us replace $l = \frac{\epsilon_2}{\sqrt{2}}$ and $k = 2$ in Eq. (10). Thus, the time to transmit the message of length L at each hop is lower bounded by $\frac{L}{R(k)} = O(1)$ as $n \to \infty$.

Therefore, the time it takes to reach x in stage I is $O(\sqrt{n}) \cdot O(1) = O(\sqrt{n})$ in $n \to \infty$. ■

Lemma 6: The time spent in Stage I and III is $O(log(n)^\alpha)$.

Proof: In Stage I, if $s = s'$, then the transmission time is zero. Otherwise, since $||s - s'|| \leq \epsilon_2 log(n)$, to transmit a message of length L, the time is $t_2 \leq L/(Blog(1 + \frac{P_{max}(\epsilon_2 log(n))^\alpha}{BN_0})) = \Theta(log(n)^\alpha)/P_{max}$, as $n \to \infty$.

In Stage III, let $l = \frac{\epsilon_2 log(n)}{\sqrt{2}}$ and $k = 25$ in Eq. (10), we have $t_3 = L/R(k) = O(log(n)^\alpha)$ as $R(k) = \Omega(1)$. ■

From Lemma 5 and Lemma 6, the time spent in Stage II dominates the total diffusion time, i.e., $t = t_1 + t_2 + t_3 = O(\sqrt{n})$. Therefore, $\omega_b = sup\{\frac{\sqrt{n}}{T(n)}\} = \Omega(1)$.

It remains to be shown that the event $E(n)$ occurs with probability 1 as $n \to \infty$.

Pentrose and Pisztora (Theorem 1, Proposition 2 in [19]) proved the following result. For $\mathcal{P}_n$, a Poisson point process of unit rate on $B(n)$, with proper choice of constant $\epsilon_2^{(1)}$ there exists a *unique* connected component in $G(n, \epsilon_2^{(1)})$ that satisfies Property (P1) and (P2). Furthermore, this cluster is part of the unbounded connected component when $n \to \infty$.

To see that Property (P3) holds, let us consider the Bernoulli percolation structure on $\mathcal{Z}^2$ by partitioning the space into lattice of edge length $\epsilon_2^{(2)}/\sqrt{2}$ (the value of $\epsilon_2^{(2)}$ to be determined next). A cell is *open* if it contains at least one point and *closed* otherwise. To obtain a bond percolation model, using the method in [4], we draw a horizontal edge across half of the cells, and a vertical edge across the others, as shown on the right-hand side of

Fig. 3. An edge is open if the cell it crosses is open. By the definition of Poisson point processes, the edges are open independently with probability $p = P\{\text{a cell contains at least one point}\} = 1 - e^{(\epsilon_2^{(2)})^2/2}$. A path is a sequence of vertices and edges $\gamma = (x_1, e_1, x_2, e_2, \ldots, x_k, e_k, x_{k+1})$ such that x_i and x_{i+1} are neighbors (with Manhattan distance = 1), and e_i is the edge between x_i and x_{i+1}. The number of edges in γ is called the length of γ and is denoted by $|\gamma|$. Moreover, we will only consider simple paths for which the visited vertices are all distinct. A path is said to be open if all its edges are open.

When $p > p_c$ (p_c is the percolation threshold), percolation theory [6] states that there exists a unique infinite connected component in $\mathcal{Z}^2$. The chemical distance $D(x,y)$ between x and y in $\mathcal{Z}^2$ is defined as,

$$D(x,y) = inf_{\gamma}(|\gamma|). \tag{11}$$

where the infimum is taken on the set of open paths whose extremities are x and y. It depends only on the Bernoulli bond percolation structure.

Antal and Pisztora (Theorem 1.1 in [2]) proved that if $p > p_c$, there exists a constant $\rho = \rho(p) \in [1, +\infty)$ and two strictly positive constants A and B such that:

$$\forall y \in \mathcal{Z}^2, P\{0 \leftrightarrow y, D(0,y) > \rho||y||_1\} \leq Aexp(-B||y||_1), \tag{12}$$

where for $y = (y_1, y_2) \in \mathcal{Z}^2$, $||y||_1$ is defined as $||y||_1 = \sum_{i=1}^{2} |y_i|$.

The result implies the open path that connects a cell y to the origin cannot be excessively long. Instead, the probability that the chemical distance of a cell y connected to the origin is larger than a constant factor of their Manhattan distance decreases exponentially as the later gets larger. Due to the shift invariant property of the bond percolation structure, we have the following:

Lemma 7: If $\epsilon_2^{(2)} \geq \sqrt{-2log(1-p_c)}$, there exists a constant $\rho \in [1, +\infty]$ and two strictly positive constant A and B such that, $\forall x, y \in \mathcal{Z}^2$,

$$P\{x \leftrightarrow y, D(x,y) > \rho\sqrt{2}||y-x||\} \leq Aexp(-B||y-x||) \tag{13}$$

Proof: From Cauchy's and triangle inequality, $||y|| \leq ||y||_1 = \sum_{i=1}^{2} |y_i| \leq \sqrt{2\sum_{i=1}^{2} |y_i|^2} = \sqrt{2}||y||$ ($||\cdot||$ is the Euclidean norm).

Since $\epsilon_2^{(2)} \leq \sqrt{-2log(1-p_c)}$, it is easy to show when

$$\begin{aligned} p &= P\{\text{a cell contains at least one point}\} \\ &= 1 - e^{(\epsilon_2^{(2)})^2/2} \\ &\geq p_c. \end{aligned}$$

there exists a constant $\rho = \rho(p) \in [1, +\infty)$ and two strictly positive constants A and B such that:

$$\begin{aligned} & P\{0 \leftrightarrow y, D(x,y) > \rho\sqrt{2}||x-y||\} \\ \leq\ & P\{0 \leftrightarrow y, D(x,y) > \rho||x-y||_1\} && (||y||_1 \leq \sqrt{2}||y||) \\ \leq\ & Aexp(-B||x-y||_1) && \text{(Theorem 1.1 in [2])} \\ \leq\ & Aexp(-B||y-x||) && (||y|| \leq ||y||_1) \end{aligned}$$

Therefore, the result holds.

■

Let $\epsilon_2 := max(\epsilon_2^{(1)}, \sqrt{\frac{5}{2}}\epsilon_2^{(2)})$ and $\mathcal{X}(n)$ be the set of vertices which fall in the large connected component on $G(B(n), \epsilon_2)$. To this end, we have shown that there exists a connected "fast" backbone $\mathcal{X}(n)$ satisfying Property (P1) $\sim$ (P3) with probability 1 as $n \to \infty$.

C. Bounds for Dense Networks

In dense networks, the upper bound in Section V-A holds, i.e., $\omega_b = O(1)$. If we scale down the edge length and distances by $\frac{1}{\sqrt{n}}$, three-stage protocol still applies. In this case, $|B(n)| = 1$. Each transmission in Stage II is at constant rate while the total number of hops is $O(\sqrt{n})$. Therefore, $\omega_b = \Omega(n^{-1/2})$. However, this bound is not tight. Instead, a tight lower bound can be found by directly broadcasting the information to all receivers. Therefore, $\omega_b = \Theta(1)$. This is somewhat counter-intuitive as if we draw analogy from the physical world, the speed of acoustic wave traveling in a medium increases with the density.

D. Discussion

In implementing the three-stage information diffusion protocol, there are a few practical issues that need to resolved. First, one needs to identify the "fast" connected backbone. Second, there needs to be a signaling mechanism to determine the termination of Stage II in order to start Stage III simultaneously at each relay seed node. Otherwise, transmissions in Stage III may potentially interfere with those in Stage II, which leads to a slower diffusion rate in Stage II.

Since the percolation threshold p_c is known to be 0.5 for square bond percolation model, we can determine the value of ϵ_2 accordingly. The second issue is trickier as unless the size of network is known ahead of time or a reliable feedback mechanism is adopted, nodes cannot have a consensus on the termination of Stage II. However, this problem is less acute in applications like event query and route discovery if the requested information is stored ahead of time at the relay seeds.

VI. Conclusion

In this paper, we investigate the asymptotic properties of information dissemination in random extended and dense networks. We have formally defined two quantities and derived tight bounds as the network scales. Our key findings are, i) in both extended and dense networks, the speed of disseminating one single message is constant with respect to the node density, ii) in extended networks, it is beneficial to use multihop relay in flooding messages throughout the network and iii) multihop relay offers no benefit in broadcast capacity or diffusion speed compared to direct broadcast in dense networks.

This work serves as the first step to characterize the information dissemination process in wireless networks. There are a few questions that remain to be addressed as future work.

- Derivation of the information-theoretical upper bound of broadcast communication through multihop relays under realistic signal attenuation and fading models.
- Understanding of the energy consumption characteristics of the information dissemination process.

We also plan to extend the results to high dimension space.

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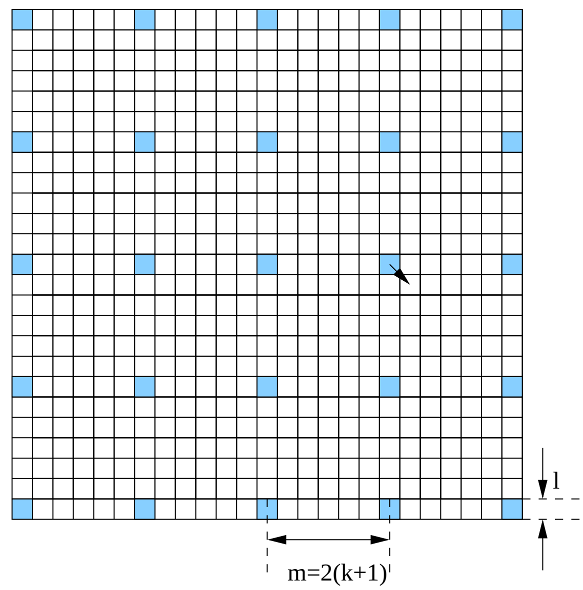

Fig. 4. One transmitter in each shaded square can transmit concurrently to receivers k squares away at rate $R(k)$. In this example, the receivers may lie in the closest diagonal square and thus $k = 2$.

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APPENDIX I
PROOF OF LEMMA 3

Lemma 3: For any given integer $k \geq 0$, under the generalized physical layer model in Section III-A, there exists a TDMA scheduling, such that one node per square of edge length l can transmit to nodes located within a radius of k squares (in Manhattan distance) with fixed rate $R(k)$ given as follows.

$$R(k) \geq \frac{B}{(2+2k)^2} log(1 + \frac{P_{max}}{BN_0(l(k+1))^\alpha + K' P_{max}}),$$

where K' is a constant independent of k and l.

Proof: Devise a TDMA schedule by partitioning the area into m^2 disjoint equivalent subsets, where $m = 2(k+1)$. As shown in Figure 4 in each slot, only shaded squares that are separated by multiples of Manhattan distance m can transmit simultaneously. Consider one transmitter u that is broadcasting to all nodes within Manhattan distance k squares, or equivalently, radius of at most $l(k+1)$ in Euclidean distance.

To find an upper bound to the interferences, we observe that the closest 8 interferers are at least $k+2$ squares or equivalently, $l(k+1)$ away in Euclidean distance. Similarly, the next 16 interferers are at least $3d+4$ squares or equivalently, $3l(k+1)$ away in Euclidean distance. Therefore, the total interferences $I(k)$ is upper bounded by,

$$\begin{aligned} I(k) &\leq \sum_{i=1}^{\infty} 8iP_{max}[l(2i-1)(k+1)]^{-\alpha} \\ &= 8P_{max}(l(k+1))^{-\alpha}\sum_{i=1}^{\infty} i(2i-1)^{-\alpha} \\ &= P_{max}(l(k+1))^{-\alpha}K', \end{aligned} \tag{14}$$

where $K' = 8\sum_{i=1}^{\infty} i(2i-1)^{-\alpha}$. As long as $\alpha > 2$, K' converges.

Since the signal strength at all receivers within the radius of at most $l(k+1)$ (denoted by $\mathcal{R}$) from the transmitter is lower bounded by $P_{max}(l(k+1))^{-\alpha}$, the achievable rate of single-hop broadcast is given by,

$$\begin{aligned} r_u &= \min_{v \in R} Blog(1 + SINR_{uv}) \\ &\geq Blog(1 + \tfrac{P_{max}(l(k+1))^{-\alpha}}{BN_0 + P_{max}(l(k+1))^{-\alpha}K'}) \\ &= Blog(1 + \tfrac{P_{max}}{BN_0 l(k+1)^{\alpha} + K'P_{max}}) \end{aligned} \tag{15}$$

To obtain the actual achievable rate $R(k)$, we need to divide r_u by m^2. So finally, $R(k) \geq \frac{B}{(2+2k)^2} log(1 + \frac{P_{max}}{BN_0 l(k+1)^{\alpha} + K'P_{max}})$

If $l \to \infty$ as $n \to \infty$, $R(k) = \Omega(l^{-\alpha})$. On the other hand, If $l \to 0$ as $n \to \infty$, $R(k) = \Omega(1)$. ■