1. Define the following concepts:
   (a) A *complete* metric space (X, ρ).
   (b) The *spectrum* of a linear bounded operator A. What is an *eigenvalue* of A?
   (c) A *complete* orthonormal set $\{\varphi_n\}$ in a Hilbert space H.
   (d) A *convex* function $f : C \to \mathbb{R}$ on an open convex $C \subset \mathbb{R}^n$.
   (e) The *synoptic* sets, the *effective domain* and the *graph* of the function $f : X \to \overline{\mathbb{R}}$. (Here, $\overline{\mathbb{R}}$ denotes the *extended real line*, i.e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.)

2. State carefully the following results, making sure that all conditions are included and significant terms are defined.
   (a) Let Ω be an open set in $\mathbb{R}^n$ and $x_0 \in \Omega$. State the inverse function theorem for a function $f : \Omega \to \mathbb{R}^n$.
   (b) State the Parseval equality for vectors in a real Hilbert space.
   (c) Let $L : H \to H$ be a continuous linear transformation on a real Hilbert space H. State the Fredholm splitting (or decomposition) theorem for H and L.
   (d) State Young's inequality for vectors in $\mathbb{R}^n$.

3. Let H be a Hilbert space; let $\mathcal{B}(H)$ denote the space of linear bounded operators from H to H; and for $A \in \mathcal{B}(H)$, let $\mathcal{R}(A)$ denote the range of A. Answer **True** or **False** to each of the following questions (work need not be shown):
   (a) If $A \in \mathcal{B}(H)$, then A is *closed*.
   (b) If M and N are sets in H such that every $x \in H$ is uniquely represented by $x = u + v$ with $u \in M$ and $v \in N$, then both M and N are *subspaces* of H.
   (c) If $A \in \mathcal{B}(H)$, then $[\ker(A^*)]^\perp = \mathcal{R}(A)$.
   (d) Let $f : X \to X$, where (X, ρ) is a metric space. If $f \circ f$ is a *contraction*, then f is *continuous*.
   (e) If a function $f : [a, b] \to \mathbb{R}$ is *convex* on $[a, b]$, then f is *uniformly continuous* on $[a, b]$.

4. Consider $f(x) := x \coth(x)$ for $x > 0$.
   (a) Find the range $R(f) \subset \mathbb{R}$ of this function and $\alpha(f) := \inf_{x>0} f(x)$. Is this infimum attained? Give reasons and proofs for your claims.
   (b) Show that f is a 1-1 map.
   (c) Let g be the inverse function of f. Prove that $g(y) < y$ for all $y \in R(f)$.

5. Let H be a real Hilbert space. Answer **True** or **False** for each of the following statements:
   (a) If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and continuous, then f has a finite lower bound on $\mathbb{R}^n$.
   (b) If $L : H \to H$ is a continuous linear transformation and $\lambda > \|L\|$, then for any $f \in H$, there is a unique solution to the equation $\lambda u - Lu = f$.
   (c) If a compact linear transformation $L : H \to H$ is 1-1 and onto, then so is its adjoint operator L^*.
   (d) If f, g are C^1 functions on the interval $(0, 1)$, then the function $h(x) := \max(f(x), g(x))$ is a C^1 function on $(0, 1)$.
   (e) If a nonempty subset E of H is an orthogonal set, then it is a linearly independent set.

6. (a) Prove that every *compact* subset K of a metric space X is *closed and bounded*.
   (b) Prove that a *closed subset* M of a *compact* metric space X is *compact*.
   (c) Prove that if X, Y are metric spaces, and $f : X \to Y$ is a continuous mapping, and K is compact in X, then the image $f(K)$ is compact in Y.

7. Consider $H = \ell^2$ with the canonical scalar product and the canonical orthonormal system of unit vectors $\{e_n\}_{n\in\mathbb{N}}$. Define the vectors $f_n := e_n - 2e_{n+2}$ for $n \in \mathbb{N}$. Thus,

$$f_1 = (1, 0, -2, 0, \dots), \quad f_2 = (0, 1, 0, -2, 0, \dots), \quad f_3 = (0, 0, 1, 0, -2, 0, \dots), \dots$$

Also, consider $S = \{f_n\}_{n\in\mathbb{N}}$.

(a) Find a finite orthonormal system $\{h_j\}_{j=1}^N$ that spans $F = S^\perp$.

(b) Define an *orthogonal projection* of x onto $M = \operatorname{span}\{e_j \mid j = 1, \dots, n\} \subset H$ by

$$P_M x := \sum_{j=1}^{n} (x, e_j)e_j.$$

Compute $P_M x$ for $x = e_1 + e_2$.

(c) Define an *orthogonal projection* of x onto F, found in part (a), by

$$P_F x := \sum_{j=1}^{N} (x, h_j)h_j.$$

Compute $P_F x$ for $x = e_1 + e_2$.

8. Show that the set S defined by

$$S = \{\varphi \mid \varphi \in C^1([0, 1]), \varphi(0) = 0, |\varphi'(x)| \le 1 \ \forall x \in [0, 1]\}.$$

is pre-compact in $C^0([0, 1])$.

9. State carefully the following theorems:

(a) The *contraction mapping* theorem.

(b) The representation theorem for a linear bounded functional on a Hilbert space.

(c) *Bessel's* inequality.

(d) Give a characterization for a *convex* function $f : I \to \mathbb{R}$ with $f \in C^1(I)$ and open $I \subset \mathbb{R}$.

(e) A characterization for a *lower semi-continuous* function $f : X \to \overline{\mathbb{R}}$ on a metric space X. (Here, $\overline{\mathbb{R}}$ denotes the *extended real line*, i.e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.)

10. Consider $f(x) := x \ln(1 + x)$ for $x > 0$.

(a) Show that this function is strictly increasing and strictly convex on $(0, \infty)$.

(b) Find $\alpha(f) := \inf_{x>0} f(x)$. Is this infimum attained?

(c) Let g be the inverse function of f. Give the domain of g and show that g is strictly increasing and strictly concave on this domain.

(d) With f, g as above, prove that $1 < g(1) < 2$.

11. (a) Let M be a closed convex set in a real Hilbert space H. Show that $y \in M$ satisfies $\rho(x, M) = \|x-y\|$ if and only if for any $z \in M$, the following inequality holds:

$$(x - y, y - z) \ge 0.$$

(b) Let two sequences $\{x_n\}$ and $\{y_n\}$ of the closed unit ball $B_1(0)$ in a Hilbert space H be such that $(x_n, y_n) \to 1$ as $n \to \infty$. Prove that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

12. Let V be a subspace of a real Hilbert space H, and let $V^\perp$ be its orthogonal complement.

(a) Show that $V^\perp$ is a closed subspace of H.

(b) Prove that if $V \subseteq W$, then $W^\perp \subseteq V^\perp$.

(c) Prove that V is dense in H if and only if $V^\perp = \{0\}$.

13. Define the following concepts:

(a) A *metric* $\rho(x, y)$ for $x, y \in X$, and *two equivalent metrics* ρ_1 and ρ_2 on X.

(b) A *compact* mapping $f : X \to Y$ where X and Y are metric spaces.

(c) A *self-adjoint* operator $S : H \to H$, *normal* operator $N : H \to H$, and *unitary* operator $U : H \to H$ on a Hilbert space H.

(d) A *weakly coercive* $f : \mathbb{R}^n \to \mathbb{R}$ and *coercive* function $g : \mathbb{R}^n \to \mathbb{R}$.

(e) A *convex* function $f : \Omega \to \overline{\mathbb{R}}$, where $\Omega \subset \mathbb{R}^n$ is a non-empty convex set.

14. Let X be a metric space. Prove the following statements:

(a) A set $F \subset X$ is closed if and only if for every convergent sequence $x_n \to x$ in X such that all $x_n \in F$, it follows that also $x \in F$.

(b) Let F be a subset of a complete metric space X. Then F is *closed* in X if and only if F (as a metric space in its own right) is *complete*.

15. Consider the sequence $\{u_n\}_{n\geq 1}$ defined by

$$u_n(x) = \cos n\pi x, \quad x \in [0, 1].$$

(a) Show that $u_n \to 0$ *weakly* in $L^2(0, 1)$. (Hint: Use the density of $C^0[0, 1]$ in $L^2(0, 1)$ and the *Weierstrass polynomial approximation theorem.*

(b) Show that the sequence $\{u_n\}_{n\geq 1}$ does not converge strongly to 0 in $L^2(0, 1)$.

16. (a) Prove that if $f \in C^0[0, 1]$, then the two-point boundary value problem

$$\begin{cases} -u'' = f \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

has a unique solution in $C^2[0, 1]$ given by

$$u(x) = \int_0^1 k(x, y) f(y)\, dy \qquad \forall x \in [0, 1],$$

with

$$k(x, y) = \begin{cases} (1 - x)y & \text{if } y \leq x \\ x(1 - y) & \text{if } x \leq y. \end{cases}$$

(b) Let us consider now the following *nonlinear* two-point boundary value problem:

$$\begin{cases} -u'' = \dfrac{u}{1 + u^2} + f \text{ in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

with f still in $C^0[0, 1]$. Using an *equivalent integral equation* formulation of (1), and the *Banach contraction mapping theorem*, prove that (1) has a unique solution in $C^2[0, 1]$.

17. Consider the function $G : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$G(x) := \|x\|_2^4 - 2\langle Ax, x\rangle,$$

where A is an $n \times n$ real symmetric matrix.

(a) Prove that this function is bounded below and has minimizers on $\mathbb{R}^n$.

(b) What is the equation satisfied by the critical points of G on $\mathbb{R}^n$?

(c) What mathematical properties can you say about the critical points and/or minimizers of G?

18. (a) Given $p \in (1, \infty)$, define the p-norm on $\mathbb{R}^n$. Write out this formula explicitly when $p = 4$.

(b) Suppose $F : X \to X$ is a function. What does it mean to say that F is a **contraction mapping**?

(c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $x \in \mathbb{R}^n$. Define the **G-derivative** of f at x.

(d) Let H be a Hilbert space, and $L : H \to H$ be a continuous linear transformation. Define the **adjoint** of L.

(e) Let H be a Hilbert space and V be a subspace of H. What is the **orthogonal complement** of V?

(f) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function. What is the definition of a **strictly convex** function?

19. State carefully the following results; making sure that all conditions are included and significant terms are defined.

(a) Let Ω be an open set in $\mathbb{R}^n$ and $x_0 \in \Omega$. State the inverse function theorem for a function $f : \Omega \to \mathbb{R}^n$.

(b) State the Riesz (or Riesz-Frechet) theorem regarding continuous linear functionals on a real Hilbert space H.

(c) Let $L : H \to H$ be a continuous linear transformation. State the Fredholm splitting (or decomposition) theorem for H and L.

(d) State the Parseval equality for vectors in a real Hilbert space.

20. Answer **True** or **False** to each of the following statements (work need not be shown):

(a) All norms in an arbitrary linear normed space X are equivalent.

(b) If M is a linear set in a Hilbert space H, then $M^{\perp\perp} = M$.

(c) If $\{\varphi_k\}$ is an orthonormal subset of a Hilbert space H, then

$$x = \sum_k (x, \varphi_k)\varphi_k \quad \text{for all } x \in H$$

is equivalent to

$$\|x\|^2 = \sum_k |(x, \varphi_k)|^2 \quad \text{for all } x \in H.$$

(d) Suppose $f : (a, b) \to \mathbb{R}$ is such that $f''(x) \geq 0$ for each $x \in (a, b)$. Then f is C^1 and convex on (a, b).

(e) There exists exactly one minimizer of a lower semi-continuous and quasi-convex function $f : \Omega \to \mathbb{R}$, where Ω is a nonempty compact convex set in $\mathbb{R}^n$.

21. Let X be a Banach space and $\|x\|$ a norm of $x \in X$. Introduce a scalar product (x, y) in X that gives rise to a new norm $\|x\|_s = \sqrt{(x, x)}$ which is, in general, different from $\|\cdot\|$. Suppose there exists a constant $\gamma > 0$ such that

$$\|x\|_s \leq \gamma\|x\| \qquad \forall x \in X.$$

Consider a linear set $M \subset X$ dense in X in metric $\|\cdot\|_s$. Suppose that for any $\hat{x} \in M$, its Fourier series $\sum_k x_k\varphi_k$, with respect to the orthogonal system $\{\varphi_k\}$, converges to $\hat{x}$ in metric $\|\cdot\|$. Prove that $\{\varphi_k\}$ is complete (or equivalently, maximal) in X in metric $\|\cdot\|_s$.

22. Define the following concepts:

(a) An *open set* $M \subset X$, where (X, ρ) is a metric space.

(b) A *complete* metric space (X, ρ).

(c) $A : D(A) \subset H \to H$ is a *closed* operator, where H is a Hilbert space.

(d) A function $f : E \to \mathbb{R}$ is *lower semi-continuous* at a point $x \in E$, where X is a metric space and $\emptyset \neq E \subset X$.

(e) A *weakly coercive* function $f : \mathbb{R}^n \to \mathbb{R}$.

23. State carefully the following theorems:

(a) The *contraction mapping* theorem.

(b) *Parseval's* identity.

(c) The *Hahn-Banach* theorem.

(d) The characterization (the necessary and sufficient conditions) of a *lower semi-continuous* function.

(e) *Second order necessary* conditions for a function $f : (a, b) \to \mathbb{R}$ to have a local minimum at $x^* \in (a, b)$.

24. (a) For the questions in this part, you may provide just a **Yes** or **No** answer without justification. Do the following linear sets of functions from $C[-, 1, 1]$ form a *subspace*?

i. monotone functions

ii. even functions

iii. polynomials

iv. polynomials of degree less than k

v. continuously differentiable functions

vi. continuous piecewise linear functions

vii. functions such that $x(0) = 0$

viii. functions such that $\int_{-1}^{1} x(t)\,dt = 0$

(b) Is the linear set $L = \{x \in \ell_2 : x = (x_1, x_2, \dots),\ \sum\limits_{k=1}^{\infty} x_k = 0\}$ a *subspace*? Explain your answer.

(c) For the questions in this part, provide a **Yes** or **No** answer with justification for a **Yes** answer and a counterexample for a **No** answer.

Given two metric spaces (X, ρ_X) and (Y, ρ_Y), let $A, B \subset X$ be two arbitrary sets such that $\rho_X(A, B) = 0$. Is it possible that $\rho_Y(f(A), f(B)) = 0$ if

i. $f : X \to Y$ is a continuous mapping;

ii. $f : X \to Y$ is a uniformly continuous mapping?

25. For $0 < \alpha \leq 1$, consider the space $C^{0,\alpha}[a, b]$ of Hölder-continuous functions $f : [a, b] \to \mathbb{R}$. Let $f \in C^{0,\alpha}[a, b]$, and define

$$\|f\|_{0,\alpha} := \inf\{L \geq 0 : |f(x) - f(y)| \leq L|x - y|^\alpha \text{ for } x, y \in [a, b]\}.$$

Also, introduce $E = \{f : [a, b] \to \mathbb{R} : f \in C^{0,\alpha}[a, b],\ f(a) = 0\}$.

(a) Show that if $f \in E$, then $|f(x) - f(y)| \leq \|f\|_{0,\alpha}|x - y|^\alpha$ for all $x, y \in [a, b]$ (i.e. the infimum in the definition of $\|f\|_{0,\alpha}$ is actually a minimum).

(b) Show that $\|f\|_{0,\alpha}$ is a norm on E.

(c) Show that $\|f\|_\infty \leq (b - a)^\alpha \|f\|_{0,\alpha}$ for $f \in E$.

(d) Show that the space $(E, \|f\|_{0,\alpha})$ is complete.

26. Let H be a (complex) Hilbert space. Suppose that $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system on H and $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of vectors in H such that

$$c^2 := \sum_{k=1}^{\infty} \|g_k\|^2 < \infty.$$

(a) Show that for every $x \in H$, the series $\sum_{k=1}^{\infty} (x, g_k)e_k$ is convergent.

(b) Using (a), define
$$Ax := \sum_{k=1}^{\infty}(x, g_k)e_k, \quad x \in H.$$
Show that A is a bounded linear operator on H, i.e. $A \in \mathcal{B}(H)$, with $\|A\| \le c$.

(c) Define $A_n \in \mathcal{B}(H)$ by
$$A_n x := \sum_{k=1}^{n}(x, g_k)e_k, \quad x \in H.$$
Show that $\|A_n - A\| \to 0$ as $n \to \infty$.

(d) Compute A^*e_n for all $n \in \mathbb{N}$, and then provide a formula for A^*x for arbitrary $x \in H$.

27. Suppose Q is an $n \times n$ positive definite matrix, A is an $m \times n$ real matrix with $\operatorname{rank} A = m$ where $m \le n - 1$, and $b \in \mathbb{R}^m$. Consider the optimization problem:
$$\text{Minimize } f(x) = (Qx, x) \quad \text{subject to } Ax = b, \ x \in \mathbb{R}^n,$$
assuming that the solution set of the linear equation has a dimension $d \ge 1$.
   (a) Give a necessary and sufficient condition for a point $x \in \mathbb{R}^n$ to be an optimal solution of this optimization problem.
   (b) Find the formula for the minimal value of this problem.
   (c) Use Lagrange multipliers to find an optimal solution to this problem for the following values:
$$Q = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 2 \end{bmatrix}, \quad b = -8.$$

28. Consider a Hilbert space H and a complete orthonormal system $\{e_n\}$ in H. Define an operator A by
$$Ax = \sum_n \zeta_n e_{n+1}, \quad \text{where} \quad H \ni x = \sum_n \zeta_n e_n \quad \text{with} \quad \|x\|^2 = \sum_n |\zeta_n|^2.$$
   (a) Show that A is *linear* and *continuous.*
   (b) What is its *adjoint* A^*?
   (c) Show that 0 may *not* be an eigenvalue of A.

29. (a) Let A be an $m \times n$ real matrix. Define the rank of A.
   (b) Suppose $F : X \to Y$ is a function with X, Y being normed vector spaces. What does it mean to say that F is **Lipschitz continuous** on X?
   (c) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Define a **subgradient** of f at x.
   (d) Let H be a Hilbert space, and $L : H \to H$ be a continuous linear transformation. Define the **adjoint** of L.
   (e) Let H be a Hilbert space and V be a subspace of H. What is the **orthogonal complement** of V?
   (f) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a function. What is the definition of a **strictly convex** function?

30. Answer T (true) or F (false) for each of the following statements.
   (a) If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and continuous, then f has a finite lower bound on $\mathbb{R}^n$.
   (b) If $L : H \to H$ is a continuous linear transformation and $\lambda > \|L\|$, then for any $f \in H$ there is a unique solution of the equation $\lambda u - Lu = f$.
   (c) If a linear transformation $L : H \to H$ is 1-1, then so also is its adjoint operator L^*.

(d) If f, g are C^1 functions on the interval $(0,1)$, then the function $h(x) := \max(f(x), g(x))$ is a C^1 function on $(0,1)$.

(e) A nonempty orthogonal subset E of a Hilbert space H is a linearly independent set.

31. Let H be a real Hilbert space, and suppose $L : H \to H$ is a continuous linear operator with $\|L\| < 1$.
    (a) Prove that the operator norm of the adjoint linear operator L^* obeys $\|L^*\| < 1$.
    (b) Prove that both $(I - L)$ and $(I - L^*)$ are 1-1 maps of H to itself.
    (c) Given $f \in H$ prove there is a unique solution of the equation $u - Lu = f$.
    (d) Find an explicit formula for the inverse operator $(I-L)^{-1}$ and find an upper bound on $\|(I-L)^{-1}\|$.

32. Let Δ_n' be the set of all probability vectors in $\mathbb{R}^n$. That is, Δ_n' is the set of vectors in $\mathbb{R}^n$ that satisfy $x_j \geq 0$ for each j and $\sum_{j=1}^n x_j = 1$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.
    (a) Give reasons why g attains both its infimum and its supremum on Δ_n'.
    (b) Describe the explicit formulae satisfied by the partial derivatives $D_j g(\hat{x})$ of g at a local minimizer of g on Δ_n'. In particular give the number of (independent) equations that must hold at a local minimizer and the number of inequalities that must hold.

33. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \|Ax - b\|_2^2$ with A being an $m \times n$ real matrix and $b \in \mathbb{R}^m$.
    (a) Show that f is a convex function on $\mathbb{R}^n$.
    (b) Find the expression for the G-derivative $\nabla f(x)$ and the equation that holds at a local minimizer of f on $\mathbb{R}^n$.
    (c) Describe conditions on A, A^T that imply a minimizer of f is actually a solution of the linear equation $Ax = b$.

34. (a) Let (X, ρ) be a complete metric space, and (Y, ρ) be a subspace of (X, ρ). Prove that (Y, ρ) is complete if and only if Y is a closed set in (X, ρ).
    (b) Let (X, ρ) be a metric space. Show that a continuous image $f(K)$ of a compact set $K \subset X$ is compact.

35. Consider a linear operator $A : X \to Y$ where X and Y are linear normed spaces. Show that A is *closed* if and only if its domain $\mathcal{D}(A)$ is a Banach space in the norm $|||x||| = \|x\|_X + \|Ax\|_Y$.

36. Let X be a linear normed space, $f \in X^*$, $f \neq 0$. Consider a hyperplane $L = \{x \in X : \langle x, f \rangle = 1\}$ (here $\langle x, f \rangle$ denotes the dual pairing of $x \in X$ and $f \in X^*$). Prove that

$$\|f\| = \frac{1}{\inf\limits_{x \in L} \|x\|}.$$

37. State carefully the following theorems:
    (a) The *Riesz Representation Theorem.*
    (b) *Bessel's* inequality.
    (c) The *Heine-Borel Theorem.*
    (d) A characterization for a *lower semi-continuous* function $f : X \to \overline{\mathbb{R}}$ on a metric space X.
    (e) *First order necessary* conditions for a function $f : \Omega \to \mathbb{R}$ to have a local minimum at $x^* \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set.

38. (a) Is $\rho(x, y) = |x - y|^2$ a metric on $\mathbb{R}$? Is the same true for $\rho(x, y) = \sqrt{|x - y|}$? Justify.
    (b) Suppose metrics ρ_1 and ρ_2 are equivalent. Show that a sequence $\{x_n\}_{n=1}^\infty \subset X$ is convergent in (X, ρ_1) if and only if it is convergent in (X, ρ_2).

(c) Show that any two of the metrics ρ_p on $\mathbb{R}^n$ are equivalent. (Editor's note: probably ρ_p (for $p \geq 1$) is meant to be the metric defined $\rho_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$ for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

39. (a) Define the space ℓ_p and its norm. For what values of p is it a Hilbert space?
   (b) For $x = (x_1, x_2, \ldots) \in \ell_2$, we set $A_n x = (x_{n+1}, x_{n+2}, \ldots)$. Show that A_n is a *linear bounded operator*, and $A_n \to 0$ *strongly* as $n \to \infty$.
   (c) Define the adjoint operator A_n^*. Then find it and investigate if it is true that $A_n^* \to 0$ strongly as $n \to \infty$.

40. Define the following concepts:
   (a) A *complete* metric space (X, ρ_X).
   (b) *Uniform* and *strong convergence* of a sequence of operators $\{T_n\}$ in the space of linear bounded operators $\mathcal{B}(X, Y)$ between normed linear spaces X and Y.
   (c) The *orthogonal complement* $M^\perp$ of a nonempty subset M in a Hilbert space H.
   (d) A *complete* orthonormal set $\{\varphi_n\}$ in a Hilbert space H.
   (e) A *convex* function $f : C \to \mathbb{R}$ on an open convex subset $C \subset \mathbb{R}^n$.

41. State carefully the following theorems:
   (a) The *Open Mapping Theorem.*
   (b) The *Contraction Mapping Theorem.*
   (c) The *Pythagoras Theorem.*
   (d) The *second order necessary condition* for a point to be a local minimizer of $f : (a, b) \to \mathbb{R}$.
   (e) A *characterization* (other than the definition) for a real-valued C^1 function f defined on an open interval I of $\mathbb{R}$ to be *convex.*

42. Answer **True** or **False** to each of the following questions (work need not be shown):
   (a) If X and Y are normed linear spaces, then strong convergence of operators $\{T_n\} \subset \mathcal{B}(X, Y)$ implies their uniform convergence.
   (b) If A is a bounded linear operator on a Hilbert space H, then $H = \operatorname{ran} A \oplus \ker A^*$.
   (c) Let $\{\varphi_n\}$ be an orthonormal system in a Hilbert space H and $\lambda = (\lambda_1, \ldots, \lambda_n, \ldots)$. Then the following assertions are equivalent:
      i. The sum $\sum_{k=1}^{\infty} \lambda_k \varphi_k$ converges in H.
      ii. $\lambda \in \ell^2$.
   (d) A contraction mapping is uniformly continuous.
   (e) There exists exactly one minimizer of a lower semi-continuous and quasi-convex function $f : \Omega \to \mathbb{R}$ where Ω is a nonempty compact convex set in $\mathbb{R}^n$.

43. Let $X := C[0, 1]$ be the usual space of continuous real-valued functions on $[0, 1]$ with the inner product

$$\langle f, g \rangle := \int_0^1 f(t) g(t) \, dt.$$

Define $\mathcal{K} : X \to X$ by $\mathcal{K} f(t) := \int_0^t f(s) \, ds$ for $f \in X$.
   (a) Show that $\mathcal{K}$ is a continuous linear transformation of X to itself and find its norm.
   (b) Define, and determine, the null space of $\mathcal{K}$.
   (c) Find the adjoint operator $\mathcal{K}^*$ (restricted to X).

(d) Does the equation $\mathcal{K}u(t) = f(t)$ have a solution in X for every $f \in X$? Give reasons for your answer.

44. Let H be a Hilbert space, and let $\mathcal{B}(H)$ denote the space of linear bounded operators from H to H. Answer **True** or **False** to each of the following questions (work need not be shown):

(a) Let $f : X \to X$ where (X, ρ) is a metric space. If the composition $f \circ f$ is a *contraction*, then f is *continuous*.

(b) The inner product of two weakly convergent sequences converges.

(c) If $A \in \mathcal{B}(H)$ is *compact*, then A^* is also a compact operator.

(d) If $f : (a, b) \to \mathbb{R}$ is such that $f''(x) \geq 0$ for each $x \in (a, b)$, then f is C^1 and convex on (a, b).

45. Consider the optimization problem:

$$\text{Minimize } f(x) = (Qx, x) \quad \text{subject to } Ax = b,\ x \in \mathbb{R}^n,$$

where Q is an $n \times n$ positive semi-definite matrix, A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, and the linear system $Ax = b$ has an $n - m$ dimensional set of solutions.

(a) Give a necessary and sufficient condition for a point $x \in \mathbb{R}^n$ to be an optimal solution of this optimization problem.

(b) Find an optimal solution to this problem for the following values:

$$Q = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 2 \end{bmatrix}, \quad b = -8.$$

46. Suppose that $f(x) = ax + bx^\beta$ for $x > 0$, with a, b, β all strictly positive.

(a) Show that f is convex and has a minimizer on $(0, \infty)$.

(b) Find the minimizer of this function and find the numbers $C > 0, \gamma$ such that

$$\inf_{x>0} f(x) = Ca^\gamma b^{1-\gamma}.$$

Verify that this C is independent of a and b.

(c) When $b, \beta > 0$ and $a = 0$, show that f is convex and bounded below but does not have a minimizer on $(0, \infty)$.

47. Let Ω be a nontrivial convex set in $\mathbb{R}^n$, (a, b) an interval in $\mathbb{R}$, and $f : \Omega \to \mathbb{R}$ a convex function with $f(\Omega) \subset (a, b)$. Show that if $\varphi : (a, b) \to \mathbb{R}$ is convex and increasing, then $g(x) := \varphi(f(x))$ is convex on Ω.

48. Consider the optimization problem:

$$\text{Minimize } f(x) = (Qx, x) - (b, x), \quad x \in \mathbb{R}^n,$$

where Q is an $n \times n$ positive semi-definite matrix, and $b \in \mathbb{R}^n$.

(a) Give a *necessary* and *sufficient* condition for a point $x \in \mathbb{R}^n$ to be an optimal solution of this optimization problem.

(b) Find an optimal solution to this problem for the following values:

$$Q = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} -2 & 2 \end{bmatrix}.$$