1. Find all solutions in $\mathbb{C}$ to the equation $z^4 = -16$.

2. Suppose that a function $f(z) = u(z) + i\,v(z)$, with u and v real-valued, is analytic in a domain D (we assume here that D is connected), and $v(z) = u^2(z)$ for every $z \in D$. Prove that f is constant on D.

3. Compute
$$\int_\gamma \frac{e^{z^3}}{z^4}\,dz,$$
where γ is a simple closed curve with $0 \notin \gamma$.

4. (TRUE or FALSE?) Let $\Delta(1)^+ = \{z \in \Delta(1) \mid \mathrm{Im}(z) > 0\}$ be the upper half disk in $\mathbb{C}$. Let f be a holomorphic function defined in $\Delta(1)^+$ and continuous on the closure $\overline{\Delta(1)^+}$. Then by using the Schwarz Reflection Principle, f can be extended holomorphically in $\Delta(1)$.

5. (TRUE or FALSE?) Let $\Omega = \Delta(0,1) - \{0\}$. Any function $f \in \mathrm{Hol}(\Omega)$ is the derivative of some other function $g \in \mathrm{Hol}(\Omega)$.

6. (TRUE or FALSE?) By the Riemann mapping theorem, any simply connected domain in $\mathbb{C}$ can be mapped by a biholomorphic map onto the unit disk.

7. (TRUE or FALSE?) There is no holomorphic function f defined on the punctured disk $\Delta(1) - \{0\}$ such that f' has a simple pole at 0.

8. Let f be a holomorphic function on a connected open set $D \subseteq \mathbb{C}$ such that $\mathrm{Re}\, f(z) = \mathrm{Im}\, f(z)$ for all $z \in D$ (i.e. its real part is equal to its imaginary part on D). Prove that f is constant. (Hint: use the Cauchy-Riemann equations.)

9. Calculate:
   (a)
$$\int_0^\infty \frac{\cos x}{x^2 + a^2}\,dx \quad (a \in \mathbb{R})$$
   (b)
$$\int_0^\infty \frac{x}{(1 + x^2)x^\alpha}\,dx \quad (0 < \alpha < 1).$$

10. (a) State Rouche's theorem.
    (b) Determine how many zeros of the polynomial $p(z) = z^5 + 3z + 1$ lie in the disk $|z| < 2$.

11. Let f be a holomorphic function defined in $\mathbb{C}$.
    (a) Suppose there exists a positive integer n such that
$$\int_{\partial\Delta(1)} \frac{f(z)}{(z-a)^n}\,dz = 0 \quad \forall a \in \Delta(1).$$
    Prove that f is a polynomial.
    (b) Suppose that for each $a \in \Delta(1)$, there exists a positive integer $n(a)$ such that
$$\int_{\partial\Delta(1)} \frac{f(z)}{(z-a)^{n(a)}}\,dz = 0.$$
    Prove that f is a polynomial.

12. Complete the following steps:
    (a) State Liouville's theorem.

(b) Compute $\oint_{\partial\Delta(0,R)} \frac{f(z)}{(z-a)(z-b)}\,dz$, where f is an entire function and $a, b \in \mathbb{C}$ with $a \neq b$.

(c) Prove Liouville's theorem using (b).

13. Integrate the following functions over $|z| = 1$:

(a) $\mathrm{Re}(z)$ (the real part of z);

(b) $\dfrac{1}{z^4}$;

(c) $z^3 \cos\left(\frac{3}{z}\right)$ (recall that $\cos z = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$);

(d) $\dfrac{1}{8z^3 - 1}$.

14. By letting $R \to \infty$, prove that

$$\lim_{R\to\infty} \int_{|z|=R} \frac{e^{1/z}}{z^k}\,dz = 0$$

for any integer $k \geq 2$.

15. (a) Write down the Cauchy-Riemann equations for an analytic function $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$.

(b) Prove that $f(z) = x^2 + y^2$ is not an analytic function.

16. (TRUE or FALSE?) Let $a \in \mathbb{C}$ be an isolated singularity of a rational function f. Then a could be an essential singularity for f.

17. (TRUE or FALSE?) Let f be a meromorphic function defined on a domain $E \subset \mathbb{C}$ with isolated singularity only. Suppose that all of the residue of f are zero. Then f must be holomorphic on E.

18. (TRUE or FALSE?) The function $\sin z$ defined on the complex plane is a bounded function.

19. Let f be a holomorphic function defined on the unit disk $\Delta(1)$ with radius of convergence 1. Prove that there is at least one point in the boundary $\partial\Delta(1)$ at which the function f cannot extend holomorphically.

20. Let $f(z) = \dfrac{1}{z^2 - 5z + 4} = \dfrac{1}{(z-4)(z-1)}$.

(a) Find the Laurent expansion of f in the annulus $\{z \mid 1 < |z| < 4\}$. Especially find a_{-1}, a_{-10}, and a_{10}.

(b) Compute $\int_\gamma f\,dz$, where γ is a positively oriented circle centered at 4 of radius 1.

21. Write i^i in the form $a + bi$.

22. Find the Laurent expansion at zero of the function $f(z) = \dfrac{2}{z^2 - 5z + 6}$ valid for $2 < |z| < 3$.

23. Show that $z^5 + 6z^3 - 10$ has exactly two zeros, counting multiplicities, in the annulus $2 < |z| < 3$.

24. (a) State Cauchy's theorem on a simply connected region.

(b) Suppose that D is simply connected and f is analytic on D. Show that there is an analytic function F on D such that $F' = f$ on D.

25. Find all solutions in $\mathbb{C}$ to the equation $z^4 = -16$.

26. (a) The function $\dfrac{z^3 - 1}{z^2 + 3z - 4}$ has a power series expansion in a neighborhood of the origin. What is its radius of convergence? Justify your assertion.

(b) Write out its power series expansion in a neighborhood of the origin.

27. (a) State the residue theorem.
    (b) Use the residue theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}.$$

28. (a) State the maximum principle for analytic functions.
    (b) Use (a) to prove the following statement (Schwarz lemma): Let $f(z)$ be holomorphic in $|z| \leq R$ with $|f(z)| \leq M$ on $|z| = R$. Then

$$|f(z) - f(0)| \leq \frac{2M|z|}{R}.$$

    (c) State and prove Liouville's theorem (about the properties of entire functions).
    (d) Deduce from Liouville's theorem that the range of a nonconstant entire function must be dense in $\mathbb{C}$.

29. Let $f(z) = \frac{z+i}{z-i}$. The lines $L_1 = \{z \mid \mathrm{Im}(z) = 1\}$, $L_2 = \{z \mid \mathrm{Im}(z) = -1\}$ and the circle $C = \{z \mid |z| = 1\}$ divide the complex space $\mathbb{C}$ into five regions. What are the corresponding regions of the image under the map f?

30. Show that the polynomial $2z^5 - 6z^2 + z + 1$ has exactly three zeros (counting multiplicities) in $\{z \mid 1 < |z| < 2\}$.

31. State and prove the Little Picard Theorem.

32. Concerning approximation of a holomorphic function by a sequence of polynomials, show that there does not exist a sequence of holomorphic polynomials $P_n(z)$ which converges to $\frac{1}{z}$ uniformly on the domain $\{z \in \mathbb{C} \mid \frac{1}{2} < |z| < \frac{3}{2}\}$.

33. Let f be holomorphic in the unit disk $\Delta(1)$ and continuous on $\overline{\Delta(1)}$. Assume that

$$|f(z)| = |e^z| \qquad \forall z \in \partial\Delta(1).$$

    Find all such f.

34. Evaluate the real integral

$$\int_0^{\infty} \frac{\log x}{1 + x^4}\, dx.$$

35. Prove the Schwarz-Pick lemma: Let $f : \Delta(1) \to \Delta(1)$ be holomorphic. Then

$$\left| \frac{f(z) - f(a)}{1 - \overline{f(a)} f(z)} \right| \leq \left| \frac{z - a}{1 - \bar{a} z} \right| \qquad \forall a, z \in \Delta(1).$$

36. Define the three types of isolated singularities (for a function f which is holomorphic on $D(a, r) \setminus \{a\}$ for some $r > 0$), and give an example for each one.

37. (a) Suppose that f can be represented by the power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ on $D(z_0, r)$. State a (integral) formula (in terms of f) to compute the coefficients a_n.
    (b) i. Let $f(z)$ be an entire function (i.e. it is holomorphic on $\mathbb{C}$) with $|f(z)| < 1$ on $\mathbb{C}$. Use the formula in (a) to prove that f is constant.
        ii. Let $f(z)$ be an entire function with $|f(z)| < 1 + |z|^{99/2}$ on $\mathbb{C}$. Use the formula in (a) to prove that f is a polynomial.

(c) Suppose that f is holomorphic on $D(0,1) \setminus \{0\}$. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of f on $D(0,1) \setminus \{0\}$. State a (integral) formula (in terms of f) to compute the coefficients a_n.

(d) Suppose that f is holomorphic on $D(0,1) \setminus \{0\}$ with $|f(z)| < 1$ for all $0 < |z| < 1$. Use the formula in (c) to prove that $a_n = 0$ for all $n < 0$, hence $z = 0$ is a removable singularity.

(e) State the (general) Riemann removable singularity theorem.

38. Determine the number of zeros (counting multiplicities) of the polynomial $z^7 - 5z + 3$ in the annulus $\{z \mid 1 < |z| < 2\}$.