1. For each of the following topological spaces X_i, determine whether X_i and $X_i \times X_i$ are homeomorphic.
   (a) $X_1 = [0, 1]$
   (b) $X_2 = \mathbb{R}^2$
   (c) $X_3 = \mathbb{Z}$
   (d) $X_4 =$ the middle-third Cantor set

2. Let $X = \mathbb{N} = \{1, 2, 3, \ldots\}$, and equip X with the topology

$$\mathcal{T} = \{U \subset X \mid (2n - 1) \in U \Rightarrow 2n \in U\}.$$

   That is, $U \in \mathcal{T}$ if and only if every odd number $(2n-1)$ that is contained in U has a successor $(2n)$ that is also contained in U. Thus $U = \{1, 2, 3, 4\} \in \mathcal{T}$ since the odd elements of U (namely 1 and 3) have successors (2 and 4) that are also contained in U; on the other hand, $V = \{1, 2, 3\} \notin \mathcal{T}$ since the odd number 3 is an element of V but its successor 4 is not.
   (a) Prove that $(X, \mathcal{T})$ is locally compact but not compact.
   (b) Determine (with proof) the connected components of $(X, \mathcal{T})$, and show that $(X, \mathcal{T})$ is locally path-connected.

3. Let $\mathbb{R}^\omega$ denote the set of all infinite sequences of real numbers and let $\mathbf{0} \in \mathbb{R}^\omega$ be the sequence of all zeros.
   (a) What is the connected component of $\mathbf{0}$ in the product topology?
   (b) What is the connected component of $\mathbf{0}$ in the uniform topology?

4. Prove that if X is Hausdorff, then any compact subset of X is closed. Also give an example of a topological space that is not Hausdorff with a compact subset that is not closed.

5. Let p be an odd prime integer. Define $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ as follows. If $m = n$, set $d(m, n) = 0$. If $m \neq n$, set $d(m, n) = 1/(r + 1)$, where r is the largest nonnegative integer such that p^r divides $m - n$.
   (a) Prove that d is a metric on $\mathbb{Z}$.
   (b) With respect to the topology on $\mathbb{Z}$ induced by the metric d, is the set of even integers closed?

6. (a) Let X be a path connected topological space and let A be a path connected subset of X. Suppose there exists a continuous map $r : X \to A$ such that $r(a) = a$ for every $a \in A$. Prove that $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.
   (b) Let D^2 denote the closed unit disk in $\mathbb{R}^2$ and notice that the unit circle $\mathbb{S}^1$ forms the boundary of D^2. Prove that there does not exist a continuous map $r : D^2 \to \mathbb{S}^1$ such that $r(z) = z$ for every $z \in \mathbb{S}^1$.

7. Let $\mathbb{R}^\omega$ denote the set of all infinite sequences of real numbers and let $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ be the set of all sequences that are eventually 0: that is, $(x_1, x_2, \ldots) \in \mathbb{R}^\infty$ if and only if there is $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N$. Determine the closure of $\mathbb{R}^\infty$ in the product topology, in the box topology, and in the uniform topology.

8. Let $L \subset \mathbb{R}^2$ be the x-axis and $H = \{(x_1, x_2) \mid x_2 > 0\}$ the upper half-plane. Let $X = H \cup L$. Given $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, let $B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^2 \mid d(\mathbf{x}, \mathbf{y}) < r\}$, where d is the usual Euclidean metric. Let

$$\mathcal{B}_1 = \{B(\mathbf{x}, r) \mid \mathbf{x} = (x_1, x_2) \in H,\ 0 < r < x_2\}.$$

   Given $x \in \mathbb{R}$ and $r > 0$, let $A(x, r) = B((x, r), r) \cup \{(x, 0)\}$. Let

$$\mathcal{B}_2 = \{A(x, r) \mid x \in \mathbb{R},\ r > 0\}.$$

   (a) Show that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for a topology on X.

(b) Determine (with proof) whether or not the topology $\mathcal{T}$ generated by $\mathcal{B}$ is first-countable and/or second-countable.

(c) Show that $(X, \mathcal{T})$ is regular.

9. Let M be a smooth manifold and $x \in M$.

(a) Define the tangent space $T_x M$ and explain why it is a vector space.

(b) Define the tangent bundle TM and explain why it is a smooth manifold.

10. Describe (with justification) the fundamental group of:

(a) the 2-sphere S^2;

(b) the 2-torus $\mathbb{T}^2$;

(c) the real projective plane $\mathbb{R}P^2$.

11. Let θ and γ be smooth 3-forms on $\mathbb{S}^7$. Prove that

$$\int_{\mathbb{S}^7} \theta \wedge d\gamma = \int_{\mathbb{S}^7} d\theta \wedge \gamma.$$

Hint: recall that if ω is a smooth k-form and η is a smooth l-form, we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

12. (a) State the Sard theorem.

(b) Let $f : \mathbb{S}^1 \to \mathbb{S}^2$ be a smooth map. Prove that f cannot be surjective.

(c) For a plane P in $\mathbb{R}^3$, let $\pi_P : \mathbb{R}^3 \to P$ denote the orthogonal projection onto P. Suppose that $g : \mathbb{S}^1 \to \mathbb{R}^3$ is a smooth embedding. Prove that there exists a plane P for which $\pi_P \circ g$ is an immersion.

13. (a) Prove that if X is a Hausdorff space, then for every point $x \in X$ and every compact subset $A \subset X$, there are disjoint neighbourhoods $U \ni x$ and $V \supset A$.

(b) Let X be a locally compact Hausdorff space and give the definition of the one-point compactification X^* (you must define both the set and the topology).

(c) Use the previous two parts to prove that every locally compact Hausdorff space is regular.

14. Let M be a smooth manifold and $S \subset M$ an embedded smooth submanifold. Let $p \in S$ and $v \in T_p M$ be such that $vf = 0$ for every $f \in C^\infty(M)$ with $f|_S \equiv 0$. Let $\iota : S \to M$ be the inclusion map and show that $v \in d\iota_p(T_p S)$. Show that this may fail if S is only assumed to be immersed (instead of embedded).

15. Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^2$ given by $F(x, y, z) = (z^2 - xy, x^2 + y^2)$.

(a) Find all the critical points of F.

(b) Determine all values (a, b) such that $F^{-1}(a, b)$ is a smooth one-dimensional submanifold of $\mathbb{R}^3$.

16. Let $[0, 1]$ have the usual topology. Let $\mathbb{R}^+$ denote the nonnegative reals, and define $X := \prod_{\alpha \in \mathbb{R}^+} [0, 1]$ with the product topology. Prove that X is not first countable. (Hint: Let $A := \{(x_\alpha) : a_\alpha = 1/2 \text{ for all but finitely many } \alpha\}$. Prove that if $\mathbf{0}$ is the tuple in X with all entries equal to 0, then $\mathbf{0} \in \overline{A}$, but no sequence of points in A converges to $\mathbf{0}$.)

17. Let X be a nonempty compact Hausdorff space.

(a) Prove that X is normal.

(b) State the Tietze extension theorem.

(c) Prove that if X is also connected, then either X consists of a single point or X is uncountable.

18. For $n \in \mathbb{N}$, let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$.
    (a) Prove that $\mathbb{S}^n$ is connected and compact for every $n \in \mathbb{N}$.
    (b) Let $\mathbb{R}^\infty$ be the space of sequences $(x_i)_{i=1}^\infty$ of real numbers such that at most finitely many of the x_i are nonzero. Embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ via $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$, we may view $\mathbb{R}^\infty$ as the union of the $\mathbb{R}^n$ as n ranges over $\mathbb{N}$. Define a topology on $\mathbb{R}^\infty$ by declaring that a set $C \subset \mathbb{R}^\infty$ is closed if and only if $C \cap \mathbb{R}^n$ is closed in $\mathbb{R}^n$ for every $n \in \mathbb{N}$. Now let $\mathbb{S}^\infty$ be the subset of $\mathbb{R}^\infty$ consisting of the union of the $\mathbb{S}^n$ as n ranges over $\mathbb{N}$. Prove that $\mathbb{S}^\infty$ is connected but not compact in $\mathbb{R}^\infty$.

19. Let $\mathbb{R}_\ell$ be the real line with the lower limit topology; that is, the topology generated by the basis $\{[a, b) \mid a < b \in \mathbb{R}\}$. Is $\mathbb{R}_\ell$ first countable? Is it second countable?

20. Consider the 2-form $\omega = z\,dx \wedge dy + (1 - 2y^2z^2)\,dy \wedge dz$ on $\mathbb{R}^3$, where we use the standard (x, y, z)-coordinates.
    (a) Let $D = \{(s, t) \in \mathbb{R}^2 \mid s^2 + t^2 \leq 1\}$ be the unit disc in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be given by $f(s, t) = (1 - s^2 - t^2)s^2$, so that $f = 0$ on ∂D. Let $F : D \to \mathbb{R}^3$ be given by
    $$F(s, t) = (f(s, t), s, t).$$
    Then $M = F(D)$ is a smooth submanifold (with boundary) of $\mathbb{R}^3$, and ∂M is the unit circle in the yz-plane. Equip M with the orientation such that F is a smooth orientation-preserving map, and compute $\int_M \omega$.
    (b) Let S^2 be the unit sphere in $\mathbb{R}^3$ with the usual orientation, and compute $\int_{S^2} \omega$.

21. Prove that no two of $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ are homeomorphic (when equipped with the standard metric topology).

22. Consider the equivalence relation on $\mathbb{C}^2 \setminus \{\mathbf{0}\}$ given by putting $(z_1, z_2) \sim (\omega z_1, \omega z_2)$ for every $\omega \in \mathbb{C} \setminus \{0\}$; write $[z_1, z_2] = \{(\omega z_1, \omega z_2) \mid \omega \in \mathbb{C} \setminus \{0\}\}$ for the equivalence class of (z_1, z_2). Recall that the complex projective plane $\mathbb{C}P^1$ is defined as the quotient space of $\mathbb{C}^2 \setminus \{\mathbf{0}\}$ by this equivalence relation, so that the elements of $\mathbb{C}P^1$ are the equivalence classes $[z_1, z_2]$.
    (a) Determine (with proof) the fundamental group of $\mathbb{C}P^1$.
    (b) Let p be a polynomial in one variable with complex coefficients, and let $G : \mathbb{C} \to \mathbb{C}P^1$ be given by $G(z) = [z, 1]$. Show that there is a unique continuous map $\tilde{p} : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $\tilde{p} \circ G = G \circ p$; that is, the diagram below commutes.
    $$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{C}P^1 \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \tilde{p}} \\ \mathbb{C} & \xrightarrow{G} & \mathbb{C}P^1 \end{array}$$
    (c) Show that the map $\tilde{p}$ is smooth when $\mathbb{C}P^1$ is given the standard smooth structure (as a real manifold).

23. Let X and Y be topological spaces, and suppose that $f : X \to Y$ is continuous and injective.
    (a) If X is Hausdorff, is it necessarily true that Y is Hausdorff? If you answer YES, provide a proof. If you answer NO, provide a counterexample.
    (b) If Y is Hausdorff, is it necessarily true that X is Hausdorff? If you answer YES, provide a proof. If you answer NO, provide a counterexample.

24. Let X and Y be topological spaces. We say that a function $f : X \to Y$ is an *open map* if whenever U is an open subset of X, then $f(U)$ is an open subset of Y. Prove that if X is compact, Y is Hausdorff and connected, and $f : X \to Y$ is a continuous open map, then f is surjective.

25. Let M be a smooth manifold and fix $p \in M$. Recall that a tangent vector $v \in T_pM$ can be viewed either as a derivation or as an equivalence class of curves. Make each of these precise (define "derivation" and "equivalence class of curves" in this setting), and describe the relationship between the two: given a derivation, state which family of curves it corresponds to, and vice versa.

26. Let G be a Lie group with identity element e. Given $v \in T_eG$, show that there is a unique left-invariant vector field X on G such that $X_e = v$. In addition, prove that X is smooth.

27. Let $\mathbb{R}_\ell$ be the real line with the lower limit topology; that is, the topology generated by the basis $\{[a, b) \mid a < b \in \mathbb{R}\}$.
    (a) Is $\mathbb{R}_\ell$ first countable? Is it second countable?
    (b) Let L be a line in the plane equipped with the subspace topology it inherits as a subset of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Are all of the resulting topological spaces homeomorphic to each other? That is, if L, L' are two such lines, is L homeomorphic to L'? If so, prove it; if not, describe all the possible topologies on L.

28. Let M be a smooth manifold and ω a differential form on M. Prove that if ω has even degree then $\omega \wedge d\omega$ is exact.

29. Let G be the Heisenberg group; that is, $G = \mathbb{R}^3$ with multiplication given by identifying (x, y, z) with the matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, so $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy)$; write $\mathbf{0}$ for the identity element with $x = y = z = 0$. Let X, Y, Z be the left invariant vector fields which evaluate at the identity to $X_\mathbf{0} = \frac{\partial}{\partial x}|_\mathbf{0}$, $Y_\mathbf{0} = \frac{\partial}{\partial y}|_\mathbf{0}$, and $Z_\mathbf{0} = \frac{\partial}{\partial z}|_\mathbf{0}$. Let $g = (a, b, c) \in G$ be an arbitrary element of G, and determine X_g, Y_g, Z_g.

30. The Klein bottle $\mathbb{K}$ is the quotient space obtained by starting with the unit square

$$\{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$$

and then making the identifications $(0, y) \sim (1, 1-y)$ for all $y \in [0, 1]$ and $(x, 0) \sim (x, 1)$ for all $x \in [0, 1]$. Use the Seifert-van Kampen theorem to compute the fundamental group of $\mathbb{K}$.

31. Let D^2 denote the closed unit disk in $\mathbb{R}^2$. Let $\mathbf{v} : D^2 \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be a continuous, nonvanishing vector field on D^2. Prove that there exists a point $z \in \mathbb{S}^1$ at which $\mathbf{v}(z)$ points directly inward. Hint: argue by contradiction.

32. Let $\mathbf{v} \in \mathbb{R}^n$ be a nonzero vector. For $c \in \mathbb{R}$, define

$$L_c = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \langle \mathbf{x}, \mathbf{v} \rangle^2 = \|\mathbf{y}\|^2 + c.\}$$

For $c \neq 0$, show that L_c is an embedded submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ of codimension 1. Here $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$.

33. Let (s, t) be coordinates on $\mathbb{R}^2$ and let (x, y, z) be coordinates on $\mathbb{R}^3$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(s, t) = (\sin(t), st^2, s^3 - 1).$$

    (a) Let X_p be the tangent vector in $T_p\mathbb{R}^2$ defined by $X_p = \frac{\partial}{\partial s}|_p - \frac{\partial}{\partial t}|_p$. Compute the push-forward $f_* X_p$.
    (b) Let ω be the smooth 1-form on $\mathbb{R}^3$ defined by $\omega = dx + x\,dy + y^2\,dz$. Compute the pullback $f^*\omega$.

34. Let X and Y be topological spaces and let $f : X \to Y$ be a map. Prove that f is continuous if and only if for every $x \in X$ and every net (z_α) such that (z_α) converges to x, we have that $(f(z_\alpha))$ converges to $f(x)$.

35. Recall that a topological space Y is said to be locally compact if for every $y \in Y$, there exists an open neighborhood U_y of y such that $\overline{U_y}$ is compact.
    (a) Give the definition of a *second countable* topological space.
    (b) Let X be a second countable, locally compact, Hausdorff space. Let $X^+ = X \cup \{\infty\}$ be the one-point compactification of X. Recall that a set V is open in X^+ if and only if V is open in X or $V = X^+ \setminus C$ for some compact set $C \subset X$. Prove that X^+ is second countable.

36. Let X be a topological space and let $A \subset X$. A retraction $r : X \to A$ is a map such that $r(x) = x$ for all $x \in A$.
    (a) State Stokes' theorem for smooth orientable manifolds with boundary.
    (b) Let M be a smooth n-dimensional compact connected orientable manifold with boundary. Prove that there exists no smooth retraction $r : M \to \partial M$. Hint: proceed by contradiction and consider a nonvanishing smooth $(n-1)$-form on ∂M.

37. Let $S^1 \subset \mathbb{C}$ be the unit circle. Let $X = \mathbb{R} \times S^1$ and $Y = \mathbb{T}^2 = S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$. Define $p : X \to Y$ by $p(x, z) = (e^{2\pi i x}, z^3)$. Pick a base point $\mathbf{x}_0 \in X$ and let $\mathbf{y}_0 = p(\mathbf{x}_0) \in Y$.
    (a) Determine the fundamental groups $\pi_1(X, \mathbf{x}_0)$ and $\pi_1(Y, \mathbf{y}_0)$.
    (b) Determine the subgroup $p_*(\pi_1(X, \mathbf{x}_0)) \subset \pi_1(Y, \mathbf{y}_0)$.

38. Let $\mathfrak{g}$ and $\mathfrak{h}$ be non-abelian two-dimensional Lie algebras. Prove that $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.

39. Consider the smooth map $F : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$F(\theta, \phi) = ((2 + \cos\phi)\cos\theta, (2 + \cos\phi)\sin\theta, \sin\phi).$$

    Let $M = F(\mathbb{R}^2)$ be the 2-torus obtained as the image of F and endowed with the orientation that makes F orientation-preserving. Consider the 2-form $\omega = x\, dy \wedge dz$. Compute $F^*\omega$ and use this to compute $\int_M \omega$. Use your answer to determine the volume of the region in $\mathbb{R}^3$ enclosed by M.

40. Give $[0, 1]$ the usual topology. Let X be a product of uncountably many copies of $[0, 1]$; view X as the set of tuples (x_α), where α ranges over the nonnegative reals $\mathbb{R}^+$ and $x_\alpha \in [0, 1]$ for all $\alpha \in \mathbb{R}^+$. Give X the product topology. Prove that X is not first countable as follows.
    (a) Let $A \subset X$ be the set of tuples (x_α) such that $x_\alpha = 1/2$ for all but finitely many values of α. Let $\mathbf{0}$ denote the tuple in X with all entries equal to 0. Prove that $\mathbf{0} \in \overline{A}$.
    (b) Prove that no sequence in A converges to $\mathbf{0}$.

41. (a) State the Urysohn lemma.
    (b) Let X be a normal topological space. Suppose that $X = V \cup W$, where V and W are open in X. Prove that there exist open sets V_1 and W_1 such that $\overline{V}_1 \subset V$, $\overline{W}_1 \subset W$, and $X = V_1 \cup W_1$.

42. Describe the universal cover of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, together with the corresponding covering map. (If you prefer, you can work with $\mathbb{C} \setminus \{0\}$.) Prove that the covering space you give is the universal cover.

43. Let X be a set with the finite complement topology (i.e. $U \subseteq X$ is open if and only if U is empty or $X \setminus U$ is finite). Exactly which subsets of X are compact? Give an argument proving that your answer is correct.

44. For each of the following topological spaces X_i, determine whether X_i and $X_i \times X_i$ are homeomorphic. Give complete proofs.
    (a) $X_1 = \mathbb{R}$.
    (b) $X_2 = \mathbb{R}^2$.
    (c) $X_3 = \mathbb{Z}$.

(d) $X_4 = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

45. Given $n \in \mathbb{N}$ and $1 \leq k \leq n$, recall that $G_k\mathbb{R}^n$ is the Grassmannian manifold consisting of the set of k-dimensional subspaces of $\mathbb{R}^n$, endowed with the usual smooth structure. Determine $\dim(G_k\mathbb{R}^n)$ and prove that $G_k\mathbb{R}^n$ is compact.

46. Let $X_1 \supset X_2 \supset X_3 \supset \cdots$ be a nested sequence of nonempty compact connected subsets of $\mathbb{R}^n$. Prove that the intersection
$$X = \bigcap_{i=1}^{\infty} X_i$$
is nonempty, compact, and connected.

47. Let A be an annulus bounded by inner circle C_1 and outer circle C_2. Define a quotient space Q by starting with A, identifying antipodal points on C_2, and then identifying points on C_1 that differ by $2\pi/3$ radians. Use the Seifert-van Kampen theorem to compute the fundamental group $\pi_1(Q)$.

48. Let G be a Lie group with multiplication $m : G \times G \to G$ defined by $m(g,h) = gh$ and inversion $\mathrm{inv} : G \to G$ defined by $\mathrm{inv}(g) = g^{-1}$. Let e denote the identity element of G.
    (a) Show that the push-forward map $m_* : T_eG \oplus T_eG \to T_eG$ is given by $m_*(X, Y) = X + Y$.
    (b) Show that the push-forward map $\mathrm{inv}_* : T_eG \to T_eG$ is given by $\mathrm{inv}_*(X) = -X$.
    (c) Show that $m : G \times G \to G$ is a submersion.

49. Give $\mathbb{R}^2$ the usual topology, and define
$$K := \{(x, y) \in \mathbb{R}^2 : x \text{ and } y \text{ are either both rational or both irrational}\}.$$
Prove that K is a connected subset of $\mathbb{R}^2$.

50. Prove or disprove: A topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\} \subset X \times X$ is closed.

51. Consider the smooth map $F : \mathbb{R}^2 \to \mathbb{R}^3$ given by
$$F(\theta, \phi) = ((2 + \cos\phi)\cos\theta, (2 + \cos\phi)\sin\theta, \sin\phi).$$
Let $M = F(\mathbb{R}^2)$ be the 2-torus obtained as the image of F and endowed with the orientation that makes F orientation-preserving. Consider the 2-form $\omega = x^2\, dy \wedge dz$. Compute $F^*\omega$ and use this to compute $\int_M \omega$.

52. Determine (with justification) whether or not each of the following smooth maps is an immersion, a submersion, an embedding, and/or a covering map. If it is a covering map, determine the degree of the covering.
    (a) $F : S^1 \to \mathbb{R}$ given by $F(x, y) = y$, where $S^1 \subset \mathbb{R}^2$ is the unit circle.
    (b) $G : S^2 \to \mathbb{R}P^2$ given by $G(x) = [x]$, where we recall that $\mathbb{R}P^2$ can be defined as the quotient space $S^2/\sim$ under the equivalence relation $x \sim -x$, and $[x] \in \mathbb{R}P^2$ is the equivalence class of $x \in S^2$. (We think of S^2 as the unit sphere in $\mathbb{R}^3$.)
    (c) $H : \mathbb{R}/\mathbb{Z} \to S^2$ given by $H([t]) = (\cos 2\pi t, \sin 2\pi t, 0)$.