Preliminary Examination in Algebra
January 2022

Instructions: This exam is 3 hours long, and you are expected to answer all problems. Calculators, books, and notes are not permitted.

Problem 1: Construct two non-isomorphic, non-Abelian groups of order 27. Prove that they are non-Abelian and non-isomorphic to one another.

Problem 2:

(a) Let p be a prime number, let G be a finite Abelian group, and let H be a subgroup of G. Prove that H has index p in G if and only if it is the kernel of surjective homomorphism

$$\varphi : G \to \mathbb{Z}/p\mathbb{Z}.$$

(b) Give an example to show that the conclusion of (a) is not true in general if G is not assumed to be Abelian.

(c) Using part (a), calculate the number of subgroups of index 5 in the group

$$G = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}.$$

Problem 3:

(a) Let R be the subring of $\mathbb{C}$ defined by

$$R = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\},$$

and let $I \subseteq R$ be the principal ideal $I = (1 + \sqrt{-5})$. Find a complete set of distinct representatives for the quotient ring R/I.

(b) Prove that the ideal I from part (a) is not a prime ideal.

Problem 4:

(a) Let p be a prime number with $p = 1 \bmod 4$. Prove that there are exactly two solutions to the equation $x^2 = -1 \bmod p$. You may use the fact that $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.

(b) Suppose that $N = p_1 p_2 \cdots p_k$, where $p_1 < \cdots < p_k$ are prime numbers satisfying $p_i = 1 \bmod 4$ for each $1 \leq i \leq k$. Calculate the number of elements of order 4 in $(\mathbb{Z}/N\mathbb{Z})^\times$.