PROBABILITY AND STATISTICS PRELIMINARY EXAM.

THERE ARE 5 QUESTIONS. ATTEMPT 5 OUT OF 5 QUESTIONS.

SHOW ALL WORKING.

THIS IS A THREE HOUR CLOSED BOOK EXAM.

EACH QUESTION IS WORTH 20 POINTS.

SHOW ALL WORKING.

GOOD LUCK.

(1) (a) Consider a homogeneous Markov chain $\{X_n\}$ with state space $\{0, 1, 2, 3\}$ and transition probabilities $p_{ij} = P(X_{n+1} = j | X_n = i)$ given by $p_{00} = 1, p_{10} = 0.1, p_{11} = 0.5, p_{12} = 0.4, p_{21} = 0.5, p_{22} = 0.3, p_{23} = 0.2, p_{33} = 1$.

(i) Determine the probability that a Markov chain starting in state 1 (i.e. $X_0 = 1$) is absorbed in state 3.

(ii) Determine the expected time to absorption from state 2 i.e. the expected time for a Markox chain starting in state 2 ($X_0 = 2$) to enter either of the absorbing states $\{0\}$ or $\{3\}$.

(b) Consider a Markov chain $\{X_n\}$ with state space $\{0, 1\}$ and transition matrix

$$P = \begin{pmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{pmatrix}$$

Find (via a stationary distribution or otherwise)

$$\lim_{n \to \infty} P(X_n = 1 | X_0 = 0)$$

and

$$\lim_{n \to \infty} P(X_n = 0, X_2 = 1, X_1 = 1 | X_0 = 0)$$

(2) Consider a random walk on the integers **Z**. If the walk is at site $x \in Z$ it moves to $x+1$ with probability p and to site $x-1$ with probability q. Let $X_n \in \mathbf{Z}$ denote the position of the walk at time n. Define

$$p_{00}^{(n)} = P\{X_n = 0, X_0 = 0\}$$

and

$$A_n = \{X_n = 0, X_0 = 0\}$$

(a) Characterize the event " a random walk starts at 0 and returns to 0 infinitely often" in terms of the sets A_n, giving a brief explanation of your characterization.

(b) Show, by using the Borel-Cantelli lemma or otherwise, that if

$$\sum_{n=0}^{\infty} p_{00}^{(n)} < \infty$$

then with probability one a random walk starting in state 0 does not return to the state 0 infinitely many times.

(c) Show that if $p \neq q$ then

$$\sum_{n=0}^{\infty} p_{00}^{(n)} < \infty$$

You may assume Stirling's approximation,

$$n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$$

(3) Suppose that X, Y are independent $N(0,1)$ random variables.

(a) Prove that the random variables (i) $X+Y$ and (ii) $X-Y$ both have distribution $N(0,2)$.

Hint: You may use the method of moment generating functions. State any theorems you use.

(b) Show that $X+Y,\ X-Y$ are independent random variables. State clearly any theorems you use.

(c) Suppose that U, V are independent and have a uniform distribution on $[0,1]$. Are $U+V, U-V$ independent? Prove your assertion.

(4) (a) Construct a sequence $\{X_n\}$ of random variables on the probability space $([0,1], \mathcal{B}, m)$ where m is Lebesgue measure with the properties

(i) $\liminf X_n = -1$

(ii) $\limsup X_n = 1$

(iii) $\lim_{n\to\infty} X_n = 0$ (in probability)

(iv) $|X_n| \leq 1$ for all n

(b) Does the sequence $\{X_n\}$ from (a) necessarily have limit 0 in the L^1 norm? In other words, need

$$\lim_n \int_0^1 |X_n| dx = 0?$$

Prove your assertion.

(5) Suppose that the random variable X is uniformly distributed on $[0,1]$ and conditional on $X = x$ the random variable Y has a uniform distribution on $(0, x)$.

(a) Find $f(x,y)$, the joint density of X and Y.

(b) Find the marginal distributions $f_Y(y)$ and $f_X(x)$.

(c) Find $P(Y > 1/2)$.