Numerical Analysis and Scientific Computing
Preprint Seria

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E. Kikinzon Yu. A. Kuznetsov Zh. Liu

Preprint #35

Department of Mathematics
University of Houston

March 2015

Monotonicity in RT_0 and PWCF methods on triangular and tetrahedral meshes

Evgeny Kikinzon* Yuri A. Kuznetsov* Zhuo Liu*

Abstract

In this paper, we derive the monotonicity conditions for condensed algebraic systems obtained by the discretization of the Poisson's problem by the classical lowest order Raviart-Thomas (RT_0) and the piecewise constant fluxes (PWCF) MFE methods on triangular and tetrahedral meshes. We also establish the correspondence between the condensed system matrices resulting from application of these two methods.

1 Introduction

In this paper, we study the monotonicity characteristics of two mixed hybrid finite element methods [2] on unstructured triangular (2D) and tetrahedral (3D) meshes. The methods we consider are the classical lowest order Raviart-Thomas (RT_0) MFE method [5], [6] and the piecewise constant fluxes (PWCF) MFE method [3], [1].

The diffusion problem we discretize using both method is as follows:

$$-\Delta p = f \quad \text{in } \Omega \tag{1}$$

with Dirichlet boundary condition:

$$p = 0 \quad \text{on } \partial\Omega\,, \tag{2}$$

where Ω is a simply connected domain either in 2D or 3D.

For the PWCF method, we derive the underlying algebraic system and show the representation of the condensed system matrices that allows to easily establish the monotonicity criteria for both triangular and tetrahedral meshes. The similar study for the RT_0 method on triangular meshes was presented in [4], we extend it to the case of

*University of Houston, Department of Mathematics, 641 PGH, Houston, TX 77204, USA. E-mail: kuz@math.uh.edu

tetrahedral meshes, and make the conclusion about the correspondence of the condensed matrices for the two methods in 2D and 3D.

In Section 2 we give the description of the mixed finite element method and the underlying algebraic system for problem (1), (2). We derive the monotonicity conditions for the PWCF method on triangular meshes in Section 3.1 and on tetrahedral meshes in Section 3.2. The monotonicity result for RT_0 MFEM on triangular meshes is given in Section 4.1, and the result on tetrahedral meshes, along with the comparison of matrices to the ones resulting from the PWCF method, is shown in Section 4.2.

2 Mixed Finite Element Method

2.1 Mixed hybrid formulation for a polygonal (2D) or polyhedral (3D) cell

Let $\boldsymbol{u} = -\nabla p$ be the flux vector function, then the equivalent mixed form of the problem (1), (2) is as follows:

$$\begin{array}{rcl} \boldsymbol{u} + \nabla p &=& 0 \quad \text{in } \Omega\,, \\ \nabla \cdot \boldsymbol{u} &=& f \quad \text{in } \Omega\,, \\ p &=& 0 \quad \text{on } \partial\Omega\,. \end{array} \tag{3}$$

The weak formulation of (3) is as follows: Find $(\boldsymbol{u}, p) \in V \times P$ such that

$$\begin{array}{rcl} \displaystyle\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx - \int_\Omega p(\nabla \cdot \boldsymbol{v})\, dx &=& 0\,, \\ \displaystyle\int_\Omega (\nabla \cdot \boldsymbol{u}) q\, dx &=& \displaystyle\int_\Omega f q\, dx \end{array} \tag{4}$$

for all $(\boldsymbol{v}, q) \in V \times P$. Here, $V = H_{div}(\Omega)$, and $P = L_2(\Omega)$.

We partition Ω into m mesh cells E_k with interfaces Γ_{kl} between mesh cells E_k and E_l, $k < l$, and faces Γ_i on the boundary $\partial\Omega$. We can write $\Omega = \sum_{k=1}^{m} E_k$. The the mixed variational macro-hybrid formulation to (3), reads as follows: find $(\overline{\boldsymbol{u}},\ \overline{p},\ \overline{\lambda}) \in V \times P \times \Lambda$ such that

$$\begin{array}{lcl} a_H(\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}}) + b_H(\overline{p}, \overline{\boldsymbol{v}}) + c_H(\overline{\lambda}, \overline{\boldsymbol{v}}) &=& 0 \\ b_H(\overline{q}, \overline{\boldsymbol{u}}) &=& l_H(\overline{q}) \\ c_H(\overline{\mu}, \overline{\boldsymbol{u}}) &=& 0 \end{array} \tag{5}$$

for all $(\overline{\boldsymbol{v}},\, \overline{q},\, \overline{\mu}) \in V \times P \times \Lambda$, where

$$
\begin{aligned}
a_H(\overline{\boldsymbol{u}},\, \overline{\boldsymbol{v}}) &= \sum_{k=1}^{m} a_{H,k}(\boldsymbol{v}_k,\, \boldsymbol{u}_k) \\
b_H(\overline{p},\, \overline{\boldsymbol{v}}) &= \sum_{k=1}^{m} b_{H,k}(p_k,\, \boldsymbol{v}_k) \\
c_H(\overline{\lambda},\, \overline{\boldsymbol{v}}) &= \sum_{\substack{k,l=1\\ k<l}}^{m} \int_{\Gamma_{kl}} (\boldsymbol{v}_k \cdot \boldsymbol{n}_k)\lambda_{kl}\, ds \\
l_H(\overline{q}) &= -\sum_{k=1}^{m} f q_k\, dx
\end{aligned}
$$

$$
a_{H,k} = \int_{E_k} \boldsymbol{u}_k \cdot \boldsymbol{v}_k\, dx\,, \quad b_{H,k} = -\int_{E_k} p_k(\nabla \cdot \boldsymbol{v}_k)\, dx\,, \quad k = 1, \ldots, m\,, \tag{6}
$$

and $\boldsymbol{n}_k$ is the outward unit normal to ∂E_k, which is the boundary of E_k, $k = 1, \ldots, m$.

Here,

$$
\begin{aligned}
\boldsymbol{V} &= V_1 \times ... \times V_m\,, \\
\boldsymbol{P} &= P_1 \times ... \times P_m\,, \\
\boldsymbol{\Lambda} &= \prod_{\substack{k,l=1\\ k<l}}^{m} \Lambda_{kl}
\end{aligned} \tag{7}
$$

with $V_k = H_{div}(E_k)$, $P_k = L_2(E_k)$, and $\Lambda_{kl} = L_2(\Gamma_{kl})$, $|\Gamma_{kl}| \neq 0$, $1 \leq k < l \leq m$.

Next, we choose finite dimensional subspaces $\boldsymbol{V}_h \subseteq \boldsymbol{V}$, $\boldsymbol{P}_h \subseteq \boldsymbol{P}$, and $\boldsymbol{\Lambda}_h \subseteq \boldsymbol{\Lambda}$. With these definitions, the mixed hybrid finite element discretization of (1), (2) reads as follows: find $(\overline{\boldsymbol{u}_h},\, \overline{p_h},\, \overline{\lambda_h}) \in V_h \times P_h \times \Lambda_h$ such that

$$
\begin{aligned}
a_H(\overline{\boldsymbol{u}}_h,\, \overline{\boldsymbol{v}}_h) \;+\; b_H(\overline{p}_h,\, \overline{\boldsymbol{v}}_h) \;+\; c_H(\overline{\lambda}_h,\, \overline{\boldsymbol{v}}_h) &= 0 \\
b_H(\overline{q}_h,\, \overline{\boldsymbol{u}}_h) \;-\; \sigma_H(\overline{p}_h,\, \overline{q}_h) &= l_H(\overline{q}_h) \\
c_H(\overline{\mu}_h,\, \overline{\boldsymbol{u}}_h) &= 0
\end{aligned} \tag{8}
$$

for all $(\overline{\boldsymbol{v}}_h, \overline{q}_h, \overline{\mu}_h) \in V_h \times P_h \times \Lambda_h$. The latter FE problem results in the system of linear algebraic equations

$$
A \begin{pmatrix} \overline{u} \\ \overline{p} \\ \overline{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{F} \\ 0 \end{pmatrix} \tag{9}
$$

with the symmetric matrix

$$
A = \begin{pmatrix} M & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{pmatrix} = \sum_{k=1}^{m} \mathcal{N}_k\, A_k\, \mathcal{N}_k^T\,, \tag{10}
$$

where

$$A_k \;=\; \begin{pmatrix} M_k & B_k^T & C_k^T \\ B_k & 0 & 0 \\ C_k & 0 & 0 \end{pmatrix}\,, \tag{11}$$

each M_k, B_k, C_k are local matrices for a cell E_k, and $\mathcal{N}_k$ is an appropriate subassembling matrix, $k = 1, \dots, m$.

2.2 Reduced algebraic system (Schur-complement)

To get the monotonicity condition for the algebraic system (9), we perform the following condensation procedure:

First, eliminating the variable $\overline{u}$, we get the system:

$$S_{p\lambda} \begin{pmatrix} \overline{p} \\ \overline{\lambda} \end{pmatrix} \;=\; \begin{pmatrix} -\overline{F} \\ 0 \end{pmatrix}\,, \tag{12}$$

where

$$S_{p\lambda} \;=\; \begin{pmatrix} B \\ C \end{pmatrix} M^{-1} \begin{pmatrix} B^T & C^T \end{pmatrix} \;=\; \sum_{k=1}^{m} \mathcal{N}_{p\lambda,k}\, S_{p\lambda,k}\, \mathcal{N}_{p\lambda,k}^T \tag{13}$$

is a symmetric positive definite (SPD) matrix.

Then, eliminating the variable $\overline{p}$, we come to the Schur-complement system:

$$S_\lambda \overline{\lambda} = \overline{\phi} \tag{14}$$

with the Schur-complement matrix

$$S_\lambda \;=\; \left(BM^{-1}B^T - BM^{-1}C^T(CM^{-1}C^T)^{-1}CM^{-1}B^T\right) \;=\; \sum_{k=1}^{m} \mathcal{N}_{\lambda,k}\, S_{\lambda,k}\, \mathcal{N}_{\lambda,k}^T \tag{15}$$

and the right-hand side

$$\overline{\phi} = C\left(M^{-1}B^T(BM^{-1}B^T)^{-1}\right)\overline{F}\,. \tag{16}$$

Here,

$$\begin{aligned} S_{p\lambda,k} \;&=\; \begin{pmatrix} B_k \\ C_k \end{pmatrix} M_k^{-1} \begin{pmatrix} B_k^T & C_k^T \end{pmatrix}\,, \\ S_{\lambda,k} \;&=\; \left(B_k M_k^{-1} B_k^T - B_k M_k^{-1} C_k^T (C_k M_k^{-1} C_k^T)^{-1} C_k M_k^{-1} B_k^T\right)\,, \end{aligned} \tag{17}$$

and $\mathcal{N}_{p\lambda,k}$ and $\mathcal{N}_{\lambda,k}$ are the appropriate subassembling matrices, $k = \overline{1,\ m}$.

3 Piece-Wise Constant Fluxes (PWCF) Method on triangular and tetrahedral meshes

3.1 Monotonicity condition for PWCF method on triangular meshes

Assume that the domain Ω is partitioned using a triangular mesh Ω_h. Let E_k be a mesh cell, we denote its faces by Γ_i, and we denote an angle opposite to Γ_i by α_i, $i = \overline{1,\ 3}$. In this Section we investigate conditions under which the global system matrix for the problem (1)–(2) is a singular M-matrix. Note that the global matrix S_λ is an M-matrix if and only if local matrices $S_{\lambda,k}$ are M-matrices for all mesh cells E_k.

We consider a triangular cell E_k. Without the loss of generality, we assume the height dropped onto the face Γ_1 to be of length 1, i.e. $h_1 = 1$. As before, we denote the angle opposite to the face Γ_i by α_i, $i = \overline{1,\ 3}$. We denote the outward unit normal vector to Γ_i by $\boldsymbol{n}_i$.

There are two distinct geometries that can be described by the measure of one of the angles, say, α_2. The first is a triangle with three acute angles, as shown on Figure 1. The second is a triangle with an obtuse angle α_2. An example of such triangle is given on Figure 2.

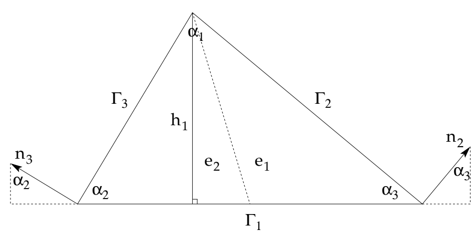

Figure 1: A triangle with three acute angles.

Regardless of the shape of α_2, we have

$$|\Gamma_1|^{(1)} = \cot\alpha_2 + \cot\alpha_3\,, \quad |\Gamma_2|^{(1)} = \frac{1}{\sin\alpha_3}\,, \quad |\Gamma_3|^{(1)} = \frac{1}{\sin\alpha_2}\,,$$

$$\boldsymbol{n}_1^{(1)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{n}_2^{(1)} = \begin{pmatrix} \sin\alpha_3 \\ \cos\alpha_3 \end{pmatrix}, \quad \boldsymbol{n}_3^{(1)} = \begin{pmatrix} -\sin\alpha_2 \\ \cos\alpha_2 \end{pmatrix}. \tag{18}$$

and

$$|E_k| = \frac{1}{2}\left(\cot\alpha_2 + \cot\alpha_3\right). \tag{19}$$

We use the PWCF method to discretize the problem (3) on E. To do so, we split the cell E_k into two triangular subcells e_1 and e_2

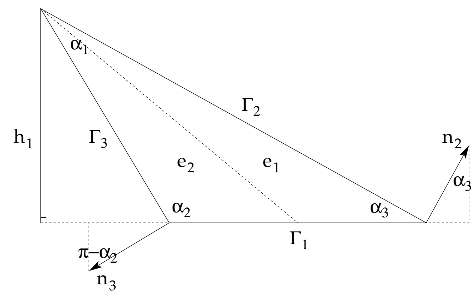

Figure 2: A triangle with an obtuse angle α_2.

by passing the line through the node opposite to the face Γ_1 and the middle of the face Γ_1. Clearly, $|e_1| = |e_2| = \frac{1}{2}|E|$.

Let $\boldsymbol{w}_i$, $i = \overline{1,\ 3}$, be the PWCF basis vector functions satisfying the following:
In e_1,

$$\begin{aligned} \boldsymbol{w}_1 \cdot \boldsymbol{n}_1 &= 1\,, \quad \boldsymbol{w}_1 \cdot n_2 = 0\,; \\ \boldsymbol{w}_2 \cdot \boldsymbol{n}_1 &= 0\,, \quad \boldsymbol{w}_2 \cdot n_2 = 1\,; \\ \boldsymbol{w}_3 &\equiv 0\,. \end{aligned}$$

In e_2,

$$\begin{aligned} \boldsymbol{w}_1 \cdot \boldsymbol{n}_1 &= 1\,, \quad \boldsymbol{w}_1 \cdot n_3 = 0\,; \\ \boldsymbol{w}_3 \cdot \boldsymbol{n}_1 &= 0\,, \quad \boldsymbol{w}_3 \cdot n_3 = 1\,; \\ \boldsymbol{w}_2 &\equiv 0\,. \end{aligned}$$

Therefore, the basis vector functions can be written explicitly:

$$\begin{aligned} \boldsymbol{w}_1 &= \begin{cases} \left(\frac{\cos\alpha_3}{\sin\alpha_3} \quad -1\right)^T & \text{in } e_1\,, \\ \left(-\frac{\cos\alpha_2}{\sin\alpha_2} \quad -1\right)^T & \text{in } e_2\,, \end{cases} \\ \boldsymbol{w}_2 &= \begin{cases} \left(\frac{1}{\sin\alpha_3} \quad 0\right)^T & \text{in } e_1\,, \\ 0\,, & \text{in } e_2\,, \end{cases} \\ \boldsymbol{w}_3 &= \begin{cases} \left(-\frac{1}{\sin\alpha_2}, 0\right)^T & \text{in } e_2\,, \\ 0\,, & \text{in } e_1\,. \end{cases} \end{aligned}$$

The resulting matrix blocks for the local system are as follows:

$$
\begin{aligned}
M_k &= \frac{|E_k|}{2} \begin{pmatrix} \dfrac{1}{\sin^2\alpha_3} + \dfrac{1}{\sin^2\alpha_2} & \dfrac{\cos\alpha_3}{\sin^2\alpha_3} & \dfrac{\cos\alpha_2}{\sin^2\alpha_2} \\ \dfrac{\cos\alpha_3}{\sin^2\alpha_3} & \dfrac{1}{\sin^2\alpha_3} & 0 \\ \dfrac{\cos\alpha_2}{\sin^2\alpha_2} & 0 & \dfrac{1}{\sin^2\alpha_2} \end{pmatrix}, \\
B_k &= -\begin{pmatrix} \cot\alpha_2 + \cot\alpha_3 & \dfrac{1}{\sin\alpha_3} & \dfrac{1}{\sin\alpha_2} \end{pmatrix}, \qquad (20) \\
C_k &= \begin{pmatrix} \cot\alpha_2 + \cot\alpha_3 & 0 & 0 \\ 0 & \dfrac{1}{\sin\alpha_3} & 0 \\ 0 & 0 & \dfrac{1}{\sin\alpha_2} \end{pmatrix}.
\end{aligned}
$$

Let

$$f_k = \frac{1}{|E_k|} \int_{E_k} f \, dx \, . \qquad (21)$$

Then, the local system for the cell E can be written as

$$\begin{pmatrix} M_k & B_k^T & C_k^T \\ B_k & 0 & 0 \\ C_k & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ p \\ \overline{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ -f_k \\ 0 \end{pmatrix} . \qquad (22)$$

Taking the Schur complement, we obtain the system in terms of p and $\overline{\lambda}$,

$$S_{p\lambda,k} \begin{pmatrix} p \\ \overline{\lambda} \end{pmatrix} = \begin{pmatrix} f_k \\ 0 \end{pmatrix} \qquad (23)$$

with the system matrix

$$S_{p\lambda,k} = \frac{1}{|E_k|} \begin{pmatrix} 4 & 0 & -2 & -2 \\ 0 & (\cot\alpha_2+\cot\alpha_3)^2 & -\cot\alpha_3(\cot\alpha_2+\cot\alpha_3) & -\cot\alpha_2(\cot\alpha_2+\cot\alpha_3) \\ -2 & -\cot\alpha_3(\cot\alpha_2+\cot\alpha_3) & 1+\frac{1}{\sin^2\alpha_3} & \cot\alpha_2\cot\alpha_3 \\ -2 & -\cot\alpha_2(\cot\alpha_2+\cot\alpha_3) & \cot\alpha_2\cot\alpha_3 & 1+\frac{1}{\sin^2\alpha_2} \end{pmatrix} \qquad (24)$$

Taking the Schur complement again, we get the system in terms of $\overline{\lambda}$,

$$S_{\lambda,k}\overline{\lambda} = \overline{F} \, , \qquad (25)$$

where

$$S_{\lambda,k} = \frac{1}{|E_k|} \begin{pmatrix} (\cot\alpha_2+\cot\alpha_3)^2 & -\cot\alpha_3(\cot\alpha_2+\cot\alpha_3) & -\cot\alpha_2(\cot\alpha_2+\cot\alpha_3) \\ -\cot\alpha_3(\cot\alpha_2+\cot\alpha_3) & \frac{1}{\sin^2\alpha_3} & \cot\alpha_2\cot\alpha_3-1 \\ -\cot\alpha_2(\cot\alpha_2+\cot\alpha_3) & \cot\alpha_2\cot\alpha_3-1 & \frac{1}{\sin^2\alpha_2} \end{pmatrix} \qquad (26)$$

and

$$\overline{F} = \frac{f_k}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} . \tag{27}$$

We assume that only α_2 can be obtuse, therefore $\cot \alpha_3 > 0$. Hence, for matrix $S_{\lambda,k}$ to be an M-matrix, we should have

$$\begin{array}{r} \alpha_3 \leq \frac{\pi}{2} \, , \\ \alpha_2 \leq \frac{\pi}{2} \, , \\ \alpha_2 + \alpha_3 \geq \frac{\pi}{2} \, . \end{array} \tag{28}$$

That condition can be written as

$$\alpha_i \leq \frac{\pi}{2} \, , \quad i = \overline{1, \ 3} \, . \tag{29}$$

Note that unlike the matrix $S_{\lambda,k}$, the matrix $S_{p\lambda,k}$ is never an M-matrix.

If condition 29 holds for all the cells E_k, $k = \overline{1, m}$, including the cells adjacent to the boundary of Ω, i.e. E_k such that $|\partial E_k \cap \partial \Omega| \neq 0$, then the global matrix S_λ is also a singular M-matrix.

3.2 Monotonicity condition for PWCF method on tetrahedral meshes

In this Section, we consider the problem (1)–(2) in the domain $\Omega \in \mathbb{R}^3$. We assume that Ω is partitioned into tetrahedral mesh cells E_k, and consider the PWCF method on the corresponding tetrahedral mesh Ω_h.

3.2.1 PWCF basis vector functions

Let E_k be a tetrahedral mesh cell with vertices V_i and faces Γ_i, $i = \overline{1, \ 4}$. We set $\Gamma_1 = (V_2V_3V_4)$, $\Gamma_2 = (V_1V_3V_4)$, $\Gamma_3 = (V_1V_2V_4)$, and $\Gamma_4 = (V_1V_2V_3)$. Let $\boldsymbol{n}_i$ be the outward unit normal vector on a face Γ_i, $i = \overline{1, \ 4}$. We partition the cell E_k into two tetrahedral subcells e_1 and e_2 with the internal triangular interface formed by the vertices V_1, V_4 and the midpoint of the edge V_2V_3, as shown on Figure 3.

We define the PWCF basis vector functions, $\boldsymbol{w}_i$, $i = \overline{1, \ 4}$, as follows: In e_1,

$$\begin{array}{lll} \boldsymbol{w}_1 \cdot \boldsymbol{n}_1 = 1 \, , & \boldsymbol{w}_1 \cdot \boldsymbol{n}_3 = 0 \, , & \boldsymbol{w}_1 \cdot \boldsymbol{n}_4 = 0 \, ; \\ \boldsymbol{w}_3 \cdot \boldsymbol{n}_1 = 0 \, , & \boldsymbol{w}_3 \cdot \boldsymbol{n}_3 = 1 \, , & \boldsymbol{w}_3 \cdot \boldsymbol{n}_4 = 0 \, ; \\ \boldsymbol{w}_4 \cdot \boldsymbol{n}_1 = 0 \, , & \boldsymbol{w}_4 \cdot \boldsymbol{n}_3 = 0 \, , & \boldsymbol{w}_4 \cdot \boldsymbol{n}_4 = 1 \, ; \\ \boldsymbol{w}_2 \equiv 0 \, . & & \end{array}$$

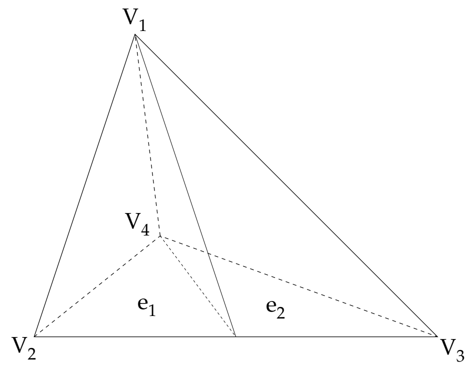

Figure 3: A tetrahedral cell E_k partitioned into subcells e_1 and e_2.

In e_2,

$$\begin{aligned}
&\boldsymbol{w}_1 \cdot \boldsymbol{n}_1 = 1\,, \quad \boldsymbol{w}_1 \cdot \boldsymbol{n}_2 = 0\,, \quad \boldsymbol{w}_1 \cdot \boldsymbol{n}_4 = 0\,;\\
&\boldsymbol{w}_2 \cdot \boldsymbol{n}_1 = 0\,, \quad \boldsymbol{w}_2 \cdot \boldsymbol{n}_2 = 1\,, \quad \boldsymbol{w}_2 \cdot \boldsymbol{n}_4 = 0\,;\\
&\boldsymbol{w}_4 \cdot \boldsymbol{n}_1 = 0\,, \quad \boldsymbol{w}_4 \cdot \boldsymbol{n}_2 = 0\,, \quad \boldsymbol{w}_4 \cdot \boldsymbol{n}_4 = 1\,;\\
&\boldsymbol{w}_3 \equiv 0\,.
\end{aligned}$$

Then, we can write the PWCF basis explicitly:

$$\begin{aligned}
\boldsymbol{w}_1 &= \begin{cases} \dfrac{\boldsymbol{n}_3 \times \boldsymbol{n}_4}{\boldsymbol{n}_1 \cdot (\boldsymbol{n}_3 \times \boldsymbol{n}_4)} & \text{in } e_1\,, \\ \dfrac{\boldsymbol{n}_2 \times \boldsymbol{n}_4}{\boldsymbol{n}_1 \cdot (\boldsymbol{n}_2 \times \boldsymbol{n}_4)} & \text{in } e_2\,, \end{cases}\\
\boldsymbol{w}_2 &= \begin{cases} 0 & \text{in } e_1\,, \\ \dfrac{\boldsymbol{n}_1 \times \boldsymbol{n}_4}{\boldsymbol{n}_2 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4)} & \text{in } e_2\,, \end{cases}\\
\boldsymbol{w}_3 &= \begin{cases} \dfrac{\boldsymbol{n}_1 \times \boldsymbol{n}_4}{\boldsymbol{n}_3 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4)} & \text{in } e_1\,, \\ 0 & \text{in } e_2\,, \end{cases}\\
\boldsymbol{w}_4 &= \begin{cases} \dfrac{\boldsymbol{n}_1 \times \boldsymbol{n}_3}{\boldsymbol{n}_4 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_3)} & \text{in } e_1\,, \\ \dfrac{\boldsymbol{n}_1 \times \boldsymbol{n}_2}{\boldsymbol{n}_4 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_2)} & \text{in } e_2\,. \end{cases}
\end{aligned} \tag{30}$$

The local matrices M_k, B_k and C_k can be written as:

$$\begin{aligned}
M_k &= \frac{|E_k|}{2} \begin{pmatrix} \|\boldsymbol{w}_1\|^2 & \boldsymbol{w}_1 \cdot \boldsymbol{w}_2 & \boldsymbol{w}_1 \cdot \boldsymbol{w}_3 & \boldsymbol{w}_1 \cdot \boldsymbol{w}_4 \\ \boldsymbol{w}_2 \cdot \boldsymbol{w}_1 & \|\boldsymbol{w}_2\|^2 & \boldsymbol{w}_2 \cdot \boldsymbol{w}_3 & \boldsymbol{w}_2 \cdot \boldsymbol{w}_4 \\ \boldsymbol{w}_3 \cdot \boldsymbol{w}_1 & \boldsymbol{w}_3 \cdot \boldsymbol{w}_2 & \|\boldsymbol{w}_3\|^2 & \boldsymbol{w}_3 \cdot \boldsymbol{w}_4 \\ \boldsymbol{w}_4 \cdot \boldsymbol{w}_1 & \boldsymbol{w}_4 \cdot \boldsymbol{w}_2 & \boldsymbol{w}_4 \cdot \boldsymbol{w}_3 & \|\boldsymbol{w}_4\|^2 \end{pmatrix}\,,\\
B_k &= \begin{pmatrix} -|\Gamma_1| & -|\Gamma_2| & -|\Gamma_3| & -|\Gamma_4| \end{pmatrix}\,,\\
C_k &= \operatorname{diag}\{|\Gamma_1|, |\Gamma_2|, |\Gamma_3|, |\Gamma_4|\}\,,
\end{aligned} \tag{31}$$

where $|E_k|$ is the volume of the tetrahedron E_k, and $|\Gamma_i|$ is the area of face Γ_i, $i = 1, ..., 4$.

By calculation, we obtain the following result:

Statement 1 *Let $S_{\lambda,k}$ be the condensed matrix defined as in (15), which is obtained by using PWCF method, then $S_{\lambda,k}$ can be represented as:*

$$S_{\lambda,k} = \frac{1}{|E_k|} \begin{pmatrix} |\Gamma_1|^2 \|\boldsymbol{n}_1\|^2 & |\Gamma_1||\Gamma_2|(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) & |\Gamma_1||\Gamma_3|(\boldsymbol{n}_1 \cdot \boldsymbol{n}_3) & |\Gamma_1||\Gamma_4|(\boldsymbol{n}_1 \cdot \boldsymbol{n}_4) \\ |\Gamma_2||\Gamma_1|(\boldsymbol{n}_2 \cdot \boldsymbol{n}_1) & |\Gamma_2|^2 \|\boldsymbol{n}_2\|^2 & |\Gamma_2||\Gamma_3|(\boldsymbol{n}_2 \cdot \boldsymbol{n}_3) & |\Gamma_2||\Gamma_4|(\boldsymbol{n}_2 \cdot \boldsymbol{n}_4) \\ |\Gamma_3||\Gamma_1|(\boldsymbol{n}_3 \cdot \boldsymbol{n}_1) & |\Gamma_3||\Gamma_2|(\boldsymbol{n}_3 \cdot \boldsymbol{n}_2) & |\Gamma_3|^2 \|\boldsymbol{n}_3\|^2 & |\Gamma_3||\Gamma_4|(\boldsymbol{n}_3 \cdot \boldsymbol{n}_4) \\ |\Gamma_4||\Gamma_1|(\boldsymbol{n}_4 \cdot \boldsymbol{n}_1) & |\Gamma_4||\Gamma_2|(\boldsymbol{n}_4 \cdot \boldsymbol{n}_2) & |\Gamma_4||\Gamma_3|(\boldsymbol{n}_4 \cdot \boldsymbol{n}_3) & |\Gamma_4|^2 \|\boldsymbol{n}_4\|^2 \end{pmatrix}. \tag{32}$$

Therefore, $S_{\lambda,k}$ is a singular M-matrix if and only if the angle between any two faces is less or equal to $\frac{\pi}{2}$.

Let us state the following facts:

Statement 2 *Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be vectors in $\mathbb{R}^3$, then*

$$(\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}))^2 = \|\boldsymbol{a}\|^2 \|\boldsymbol{b} \times \boldsymbol{c}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})\left((\boldsymbol{b} \times \boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{c})\right) + (\boldsymbol{a} \cdot \boldsymbol{c})\left((\boldsymbol{b} \times \boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b})\right). \tag{33}$$

Statement 3 *Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ be vectors in $\mathbb{R}^3$, then*

$$\begin{aligned} & (\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{d}))(\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{d})) \\ = \; & (\boldsymbol{a} \cdot \boldsymbol{b}) \|\boldsymbol{c} \times \boldsymbol{d}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{c})\left((\boldsymbol{c} \times \boldsymbol{d}) \cdot (\boldsymbol{b} \times \boldsymbol{d})\right) + (\boldsymbol{a} \cdot \boldsymbol{d})\left((\boldsymbol{c} \times \boldsymbol{d}) \cdot (\boldsymbol{b} \times \boldsymbol{c})\right) \\ = \; & (\boldsymbol{a} \cdot \boldsymbol{b}) \|\boldsymbol{c} \times \boldsymbol{d}\|^2 - (\boldsymbol{b} \cdot \boldsymbol{c})\left((\boldsymbol{c} \times \boldsymbol{d}) \cdot (\boldsymbol{a} \times \boldsymbol{d})\right) + (\boldsymbol{b} \cdot \boldsymbol{d})\left((\boldsymbol{c} \times \boldsymbol{d}) \cdot (\boldsymbol{a} \times \boldsymbol{c})\right). \end{aligned} \tag{34}$$

Then, we derive the inverse of matrix M_k, which is as follows:

$$M_k^{-1} = \frac{1}{|E_k|} \begin{pmatrix} \|\boldsymbol{n}_1\|^2 & \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 & \boldsymbol{n}_1 \cdot \boldsymbol{n}_3 & \boldsymbol{n}_1 \cdot \boldsymbol{n}_4 \\ \boldsymbol{n}_2 \cdot \boldsymbol{n}_1 & \|\boldsymbol{n}_2\|^2 + \frac{(\boldsymbol{n}_2 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4))^2}{\|\boldsymbol{n}_1 \times \boldsymbol{n}_4\|^2} & \boldsymbol{n}_2 \cdot \boldsymbol{n}_3 - \frac{(\boldsymbol{n}_2 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4))(\boldsymbol{n}_3 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4))}{\|\boldsymbol{n}_1 \times \boldsymbol{n}_4\|^2} & \boldsymbol{n}_2 \cdot \boldsymbol{n}_4 \\ \boldsymbol{n}_3 \cdot \boldsymbol{n}_1 & \boldsymbol{n}_3 \cdot \boldsymbol{n}_2 - \frac{(\boldsymbol{n}_3 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4))(\boldsymbol{n}_2 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4))}{\|\boldsymbol{n}_1 \times \boldsymbol{n}_4\|^2} & \|\boldsymbol{n}_3\|^2 + \frac{(\boldsymbol{n}_3 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4))^2}{\|\boldsymbol{n}_1 \times \boldsymbol{n}_4\|^2} & \boldsymbol{n}_3 \cdot \boldsymbol{n}_4 \\ \boldsymbol{n}_4 \cdot \boldsymbol{n}_1 & \boldsymbol{n}_4 \cdot \boldsymbol{n}_2 & \boldsymbol{n}_4 \cdot \boldsymbol{n}_3 & \|\boldsymbol{n}_4\|^2 \end{pmatrix}. \tag{35}$$

Using the Statements (2), (3), one can verify by matrix multiplication that $MM^{-1} = I_4$.

Let us state another fact used in our derivation:

Statement 4 *In the notations used, we have the following relationship:*

$$|\Gamma_2|\left(\boldsymbol{n}_2 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4)\right) = -|\Gamma_3|\left(\boldsymbol{n}_3 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4)\right) \tag{36}$$

Proof Let $\boldsymbol{u}_1 = \overrightarrow{V_1V_2}$, $\boldsymbol{u}_2 = \overrightarrow{V_1V_3}$, $\boldsymbol{u}_3 = \overrightarrow{V_1V_4}$, then we have

$$\begin{aligned} \boldsymbol{n}_1 &= \frac{(\boldsymbol{u}_3 - \boldsymbol{u}_2) \times (\boldsymbol{u}_2 - \boldsymbol{u}_1)}{|\Gamma_1|}, & \boldsymbol{n}_2 &= \frac{\boldsymbol{u}_2 \times \boldsymbol{u}_3}{|\Gamma_2|}, \\ \boldsymbol{n}_3 &= \frac{\boldsymbol{u}_3 \times \boldsymbol{u}_1}{|\Gamma_3|}, & \boldsymbol{n}_4 &= \frac{\boldsymbol{u}_1 \times \boldsymbol{u}_2}{|\Gamma_4|}. \end{aligned} \tag{37}$$

Consequently,

$$\begin{aligned}
\boldsymbol{n}_2 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4) &= \boldsymbol{n}_1 \cdot (\boldsymbol{n}_4 \times \boldsymbol{n}_2) = \boldsymbol{n}_1 \cdot \frac{(\boldsymbol{u}_1 \times \boldsymbol{u}_2) \times (\boldsymbol{u}_2 \times \boldsymbol{u}_3)}{|\Gamma_2||\Gamma_4|} \\
&= \frac{((\boldsymbol{u}_3 \times \boldsymbol{u}_1).\boldsymbol{u}_2)(\boldsymbol{u}_2 \cdot (\boldsymbol{u}_1 \times \boldsymbol{u}_3))}{|\Gamma_1||\Gamma_2||\Gamma_4|}
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
\boldsymbol{n}_3 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_4) &= \boldsymbol{n}_1 \cdot (\boldsymbol{n}_4 \times \boldsymbol{n}_3) = \boldsymbol{n}_1 \cdot \frac{(\boldsymbol{u}_1 \times \boldsymbol{u}_2) \times (\boldsymbol{u}_3 \times \boldsymbol{u}_1)}{|\Gamma_3||\Gamma_4|} \\
&= -\frac{((\boldsymbol{u}_3 \times \boldsymbol{u}_2).\boldsymbol{u}_1)(\boldsymbol{u}_1 \cdot (\boldsymbol{u}_2 \times \boldsymbol{u}_3))}{|\Gamma_1||\Gamma_3||\Gamma_4|}
\end{aligned} \tag{39}$$

The result follows.

Using the facts above, we derive

$$M^{-1}B^T(BM^{-1}B^T)^{-1}BM^{-1} =$$

$$\frac{1}{|E_k|}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{(\boldsymbol{n}_2\cdot(\boldsymbol{n}_1\times\boldsymbol{n}_4))^2}{\|\boldsymbol{n}_1\times\boldsymbol{n}_4\|^2} & -\frac{(\boldsymbol{n}_2\cdot(\boldsymbol{n}_1\times\boldsymbol{n}_4))(\boldsymbol{n}_3\cdot(\boldsymbol{n}_1\times\boldsymbol{n}_4))}{\|\boldsymbol{n}_1\times\boldsymbol{n}_4\|^2} & 0 \\ 0 & -\frac{(\boldsymbol{n}_2\cdot(\boldsymbol{n}_1\times\boldsymbol{n}_4))(\boldsymbol{n}_3\cdot(\boldsymbol{n}_1\times\boldsymbol{n}_4))}{\|\boldsymbol{n}_1\times\boldsymbol{n}_4\|^2} & \frac{\left(\boldsymbol{n}_3\cdot(\boldsymbol{n}_1\times\boldsymbol{n}_4)^2\right)}{\|\boldsymbol{n}_1\times\boldsymbol{n}_4\|^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{40}$$

The result given in Statement (1) can then be easily obtained. If the condition in the statement holds for all the cells E_k, $k = \overline{1, m}$, including the cells adjacent to the boundary of Ω, i.e. E_k such that $|\partial E_k \cap \partial\Omega| \neq 0$, then the global matrix S_λ is also a singular M-matrix.

Additionally, let

$$\begin{aligned} \alpha &= \frac{\boldsymbol{n}_2.(\boldsymbol{n}_1\times\boldsymbol{n}_4)}{||\boldsymbol{n}_1\times\boldsymbol{n}_4||}, \\ \beta &= \frac{\boldsymbol{n}_3.(\boldsymbol{n}_1\times\boldsymbol{n}_4)}{||\boldsymbol{n}_1\times\boldsymbol{n}_4||}, \end{aligned} \tag{41}$$

then we can derive

$$BM^{-1}B^T = -\frac{4\alpha\beta|\Gamma_2||\Gamma_3|}{|E_k|}, \tag{42}$$

$$BM^{-1}C^T = \frac{1}{|E_k|}\begin{pmatrix} 0 & 2\alpha\beta|\Gamma_2||\Gamma_3| & 2\alpha\beta|\Gamma_2||\Gamma_3| & 0 \end{pmatrix}, \tag{43}$$

and

$$CM^{-1}C^T =$$

$$\frac{1}{|E_k|}\begin{pmatrix} |\Gamma_1|^2\|\boldsymbol{n}_1\|^2 & |\Gamma_1||\Gamma_2|(\boldsymbol{n}_1\cdot\boldsymbol{n}_2) & |\Gamma_1||\Gamma_3|(\boldsymbol{n}_1\cdot\boldsymbol{n}_3) & |\Gamma_1||\Gamma_4|(\boldsymbol{n}_1\cdot\boldsymbol{n}_4) \\ |\Gamma_2||\Gamma_1|(\boldsymbol{n}_2\cdot\boldsymbol{n}_1) & |\Gamma_2|^2(\|\boldsymbol{n}_2\|^2+\alpha^2) & |\Gamma_2||\Gamma_3|(\boldsymbol{n}_2\cdot\boldsymbol{n}_3-\alpha\beta) & |\Gamma_2||\Gamma_4|(\boldsymbol{n}_2\boldsymbol{n}_4) \\ |\Gamma_3||\Gamma_1|(\boldsymbol{n}_3\cdot\boldsymbol{n}_1) & |\Gamma_3||\Gamma_2|(\boldsymbol{n}_3\cdot\boldsymbol{n}_2-\alpha\beta) & |\Gamma_3|^2(\|\boldsymbol{n}_3\|^2+\beta^2) & |\Gamma_3||\Gamma_4|(\boldsymbol{n}_3\cdot\boldsymbol{n}_4) \\ |\Gamma_4||\Gamma_1|(\boldsymbol{n}_4\cdot\boldsymbol{n}_1) & |\Gamma_4||\Gamma_2|(\boldsymbol{n}_4\cdot\boldsymbol{n}_2) & |\Gamma_4||\Gamma_3|(\boldsymbol{n}_4\cdot\boldsymbol{n}_3) & |\Gamma_4|^2\|\boldsymbol{n}_4\|^2 \end{pmatrix}. \tag{44}$$

This gives us the reduced matrix $S_{p\lambda,k}$, first introduced in (13).

4 Lowest order Raviart-Thomas (RT_0) Method on triangular and tetrahedral meshes

4.1 Monotonicity condition for RT_0 Method on triangular meshes

Let us consider the problem (1)–(2) in the domain $\Omega \in \mathbb{R}^2$. We assume that Ω is partitioned into triangular mesh cells E_k, and consider the RT_0 method on the corresponding triangular mesh Ω_h.

As shown in [4], the monotonicity condition for the local matrix $S_{\lambda,k}$ is as follows:
$S_{\lambda,k}$ is a singular M-matrix if and only if none of the interior angles of mesh cell E_k are obtuse angles.

Under the notations introduced in Section 3.1, it can be written as

$$\alpha_i \le \frac{\pi}{2}\,, \quad i = \overline{1,\ 3}\,. \tag{45}$$

Note that matrices S_λ and, therefore, the monotonicity conditions, coincide for RT_0 and PWCF methods on triangular meshes.

4.2 Monotonicity condition for RT_0 Method on tetrahedral meshes

In this Section, we consider the problem (1)–(2) in the domain $\Omega \in \mathbb{R}^3$. We assume that Ω is partitioned into tetrahedral mesh cells E_k, and consider the RT_0 method on the corresponding tetrahedral mesh Ω_h.

First, let us observe the following fact:

Statement 5 *The condensed matrix $S_{\lambda,k}$ is independent of basis choice on $RT_0(K)$ space.*

Proof Let $\{\boldsymbol{w}_i\}$, $\{\boldsymbol{e}_i\}$ be two sets of basis vector functions for $RT_0(K)$, then there exists a linear transformation P such that

$$P(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4) = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4). \tag{46}$$

Therefore, for matrices M, B, C for two different basis, we have

$$M_e = P^T M_w P\,, \quad B_e = B_w P\,, \quad C_e = C_w P\,. \tag{47}$$

Hence,

$$\begin{aligned} S_e &= C_e M_e^{-1}\left(M_e - B_e^T (B_e M_e^{-1} B_e^T)^{-1} B_e\right) M_e^{-1} C_e^T \\ &= C_w P P^{-1} M_w^{-1} P^{-T}\left(P^T M_w P - P^T B_w^T (B_w P P^{-1} M_w^{-1} P^{-T} P^T B_w)^{-1} B_w P\right) \cdot \\ &\quad P^{-1} M_w^{-1} P^{-T} P^T C_w^T \\ &= C_w M_w^{-1}\left(M_w - B_w^T (B_w M_w^{-1} B_w^T)^{-1} B_w\right) M_w^{-1} C_w^T \\ &= S_k. \end{aligned} \tag{48}$$

Since $S_{\lambda,k}$ is independent of basis choice on $RT_0(K)$ space, instead of using classical basis $\{\boldsymbol{w}_i\}$ for $\boldsymbol{u}_h$, s.t. $\boldsymbol{w}_i \cdot \boldsymbol{n}_j = \delta_{ij}$ on Γ_j, we use the following basis:

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_4 = \begin{pmatrix} x_1 - x_1^c \\ x_2 - x_2^c \\ x_3 - x_3^c \end{pmatrix} \tag{49}$$

where $\boldsymbol{x}^c = (x_1^c, x_2^c, x_3^c)^T$ is the barycenter of E_k, i.e.

$$\begin{aligned} x_1^c &= \frac{1}{|E_k|}\int_{E_k} x_1\,dx, \\ x_2^c &= \frac{1}{|E_k|}\int_{E_k} x_2\,dx, \\ x_3^c &= \frac{1}{|E_k|}\int_{E_k} x_3\,dx. \end{aligned} \tag{50}$$

As a result, the matrix M_k now becomes a diagonal matrix and can be easily inverted:

$$M_k = |E_k| \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{|E_k|}\int_{E_k} (x_1 - x_1^c)^2 + (x_2 - x_2^c)^2 + (x_3 - x_3^c)^2\,dx \end{pmatrix} \tag{51}$$

The corresponding matrices B_k and C_k are:

$$B_k = -\begin{pmatrix} \nabla \cdot \boldsymbol{e}_1 & \nabla \cdot \boldsymbol{e}_2 & \nabla \cdot \boldsymbol{e}_3 & \nabla \cdot \boldsymbol{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -3|E_k| \end{pmatrix}, \tag{52}$$

$$
\begin{aligned}
C_k &= \begin{pmatrix}
\int_{\Gamma_1} \boldsymbol{e}_1\cdot\boldsymbol{n}_1\,dx & \int_{\Gamma_1} \boldsymbol{e}_2\cdot\boldsymbol{n}_1\,dx & \int_{\Gamma_1} \boldsymbol{e}_3\cdot\boldsymbol{n}_1\,dx & \int_{\Gamma_1} \boldsymbol{e}_4\cdot\boldsymbol{n}_1\,dx \\
\int_{\Gamma_2} \boldsymbol{e}_1\cdot\boldsymbol{n}_2\,dx & \int_{\Gamma_2} \boldsymbol{e}_2\cdot\boldsymbol{n}_2\,dx & \int_{\Gamma_2} \boldsymbol{e}_3\cdot\boldsymbol{n}_2\,dx & \int_{\Gamma_2} \boldsymbol{e}_4\cdot\boldsymbol{n}_2\,dx \\
\int_{\Gamma_3} \boldsymbol{e}_1\cdot\boldsymbol{n}_3\,dx & \int_{\Gamma_3} \boldsymbol{e}_2\cdot\boldsymbol{n}_3\,dx & \int_{\Gamma_3} \boldsymbol{e}_3\cdot\boldsymbol{n}_3\,dx & \int_{\Gamma_3} \boldsymbol{e}_4\cdot\boldsymbol{n}_3\,dx \\
\int_{\Gamma_4} \boldsymbol{e}_1\cdot\boldsymbol{n}_4\,dx & \int_{\Gamma_4} \boldsymbol{e}_2\cdot\boldsymbol{n}_4\,dx & \int_{\Gamma_4} \boldsymbol{e}_3\cdot\boldsymbol{n}_4\,dx & \int_{\Gamma_4} \boldsymbol{e}_4\cdot\boldsymbol{n}_4\,dx
\end{pmatrix} \\
&= \begin{pmatrix}
|\Gamma_1|n_{1,x_1} & |\Gamma_1|n_{1,x_2} & |\Gamma_1|n_{1,x_3} & \int_{\Gamma_1} n_{1,x_1}(x_1-x_1^c)+n_{1,x_2}(x_2-x_2^c)+n_{1,x_3}(x_3-x_3^c)\,dx \\
|\Gamma_2|n_{2,x_1} & |\Gamma_2|n_{2,x_2} & |\Gamma_2|n_{2,x_3} & \int_{\Gamma_2} n_{2,x_1}(x_1-x_1^c)+n_{2,x_2}(x_2-x_2^c)+n_{2,x_3}(x_3-x_3^c)\,dx \\
|\Gamma_3|n_{3,x_1} & |\Gamma_3|n_{3,x_2} & |\Gamma_3|n_{3,x_3} & \int_{\Gamma_3} n_{3,x_1}(x_1-x_1^c)+n_{3,x_2}(x_2-x_2^c)+n_{3,x_3}(x_3-x_3^c)\,dx \\
|\Gamma_4|n_{4,x_1} & |\Gamma_4|n_{4,x_2} & |\Gamma_4|n_{4,x_3} & \int_{\Gamma_4} n_{4,x_1}(x_1-x_1^c)+n_{4,x_2}(x_2-x_2^c)+n_{4,x_3}(x_3-x_3^c)\,dx
\end{pmatrix},
\end{aligned} \tag{53}
$$

where $\boldsymbol{n}_i = \begin{pmatrix} n_{i,x_1} & n_{i,x_2} & n_{i,x_3} \end{pmatrix}^T$, $i = 1, 2, 3, 4$.

Simple calculations lead to:

$$
\begin{aligned}
G_k &= M_k^{-1} B_k^T = \begin{pmatrix} 0 & 0 & 0 & -\dfrac{3|E_k|}{\displaystyle\int_{E_k} (x_1 - x_1^c)^2 + (x_2 - x_2^c)^2 + (x_3 - x_3^c)^2\,dx} \end{pmatrix}^T, \\
g_k &= B_k M_k^{-1} B_k^T = \frac{9|E_k|^2}{\displaystyle\int_{E_k} (x_1 - x_1^c)^2 + (x_2 - x_2^c)^2 + (x_3 - x_3^c)^2\,dx}, \\
H_k &= M_k^{-1} - \frac{1}{g_k} G_k G_k^T = \frac{1}{|E_k|} \cdot \mathrm{diag}\{1, 1, 1, 0\},
\end{aligned} \tag{54}
$$

and the condensed matrix $S_{\lambda,k} = C_k H_k C_k^T$ with entries:

$$
(S_{\lambda,k})_{i,j} = \frac{1}{|E_k|} |\Gamma_i||\Gamma_j| \boldsymbol{n}_i \cdot \boldsymbol{n}_j, \qquad i, j = 1, 2, 3, 4. \tag{55}
$$

This matrix is exactly same as the condensed matrix in (32), therefore we can state the following:

Statement 6 *The condensed matrices $S_{\lambda,k}$ for PWCF and RT_0 methods coincide, and, therefore, the monotonicity conditions for both methods are also the same.*

Therefore, the global matrices S_λ also coincide for both methods, with monotonicity conditions being the same as derived in Section 3.2.

Now, let

$$
\eta = \int_{E_k} (x_1 - x_1^c)^2 + (x_2 - x_2^c)^2 + (x_3 - x_3^c)^2\,dx, \tag{56}
$$

then

$$
BM^{-1}B^T = \frac{9|E_k|^2}{\eta}, \tag{57}
$$

Let us first consider matrix C_k. The last column of C_k is composed of $\int_{\Gamma_i} \boldsymbol{e}_4 \cdot \boldsymbol{n}_i \, dx$, where

$$\boldsymbol{e}_4 = \begin{pmatrix} x_1 - x_1^c \\ x_2 - x_2^c \\ x_3 - x_3^c \end{pmatrix}. \tag{58}$$

Since for any points on the face Γ_i, $\boldsymbol{e}_4 \cdot \boldsymbol{n}_i = \mathrm{dist}(\boldsymbol{x}^c, \Gamma_i)$, we have that $\int_{\Gamma_i} \boldsymbol{e}_4 \cdot \boldsymbol{n}_i \, dx$ equals three times the volume of subtetrahedrons with vertices $\boldsymbol{x}^c$ and three vertexes of Γ_i. One important property for barycenter $\boldsymbol{x}^c$ is that the the volumes of four subtetrahedrons obtained by connecting $\boldsymbol{x}^c$ and four faces are equal, therefore,

$$\int_{\Gamma_1} \boldsymbol{e}_4 \cdot \boldsymbol{n}_1 \, dx = \int_{\Gamma_2} \boldsymbol{e}_4 \cdot \boldsymbol{n}_2 \, dx = \int_{\Gamma_3} \boldsymbol{e}_4 \cdot \boldsymbol{n}_3 \, dx = \int_{\Gamma_4} \boldsymbol{e}_4 \cdot \boldsymbol{n}_4 \, dx. \tag{59}$$

Another important equation is:

$$\sum_{i=1}^{4} \int_{\Gamma_i} \boldsymbol{e}_4 \cdot \boldsymbol{n}_i \, dx = \int_E \nabla \cdot \boldsymbol{e}_4 \, dx = 3|E_k|, \tag{60}$$

therefore,

$$\int_{\Gamma_i} \boldsymbol{e}_4 \cdot \boldsymbol{n}_i \, dx = \frac{3|E_k|}{4}, \quad i = 1, 2, 3, 4. \tag{61}$$

Let us calculate $S_{p,\lambda}$ as defined in (13). We have

$$BM^{-1}C^T = \begin{pmatrix} -\frac{9|E_k|^2}{4\eta} & -\frac{9|E_k|^2}{4\eta} & -\frac{9|E_k|^2}{4\eta} & -\frac{9|E_k|^2}{4\eta} \end{pmatrix}, \tag{62}$$

and

$$\begin{aligned} CM^{-1}C^T \quad = \quad & \frac{1}{|E_k|} \begin{pmatrix} |\Gamma_1|^2\|\boldsymbol{n}_1\|^2 & |\Gamma_1||\Gamma_2|(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) & |\Gamma_1||\Gamma_3|(\boldsymbol{n}_1 \cdot \boldsymbol{n}_3) & |\Gamma_1||\Gamma_4|(\boldsymbol{n}_1 \cdot \boldsymbol{n}_4) \\ |\Gamma_2||\Gamma_1|(\boldsymbol{n}_2 \cdot \boldsymbol{n}_1) & |\Gamma_2|^2\|\boldsymbol{n}_2\|^2 & |\Gamma_2||\Gamma_3|(\boldsymbol{n}_2 \cdot \boldsymbol{n}_3) & |\Gamma_2||\Gamma_4|(\boldsymbol{n}_2 \cdot \boldsymbol{n}_4) \\ |\Gamma_3||\Gamma_1|(\boldsymbol{n}_3 \cdot \boldsymbol{n}_1) & |\Gamma_3||\Gamma_2|(\boldsymbol{n}_3 \cdot \boldsymbol{n}_2) & |\Gamma_3|^2\|\boldsymbol{n}_3\|^2 & |\Gamma_3||\Gamma_4|(\boldsymbol{n}_3 \cdot \boldsymbol{n}_4) \\ |\Gamma_4||\Gamma_1|(\boldsymbol{n}_4 \cdot \boldsymbol{n}_1) & |\Gamma_4||\Gamma_2|(\boldsymbol{n}_4 \cdot \boldsymbol{n}_2) & |\Gamma_4||\Gamma_3|(\boldsymbol{n}_4 \cdot \boldsymbol{n}_3) & |\Gamma_4|^2\|\boldsymbol{n}_4\|^2 \end{pmatrix} \\ + \quad & \frac{1}{|E_k|} \begin{pmatrix} \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} \\ \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} \\ \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} \\ \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} & \frac{9|E_k|^3}{16\eta} \end{pmatrix}. \end{aligned} \tag{63}$$

We can conclude the following:

Statement 7 *Unlike the condensed matrices $S_{\lambda,k}$, the reduced matrices $S_{p\lambda,k}$ for PWCF and RT_0 methods do not coincide.*

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