LAST TIME: Cartesian oval, exact refraction at a spherical surface, refraction matrix, transfer matrix, focal length, system matrix, object-image matrix, and thin lens equation

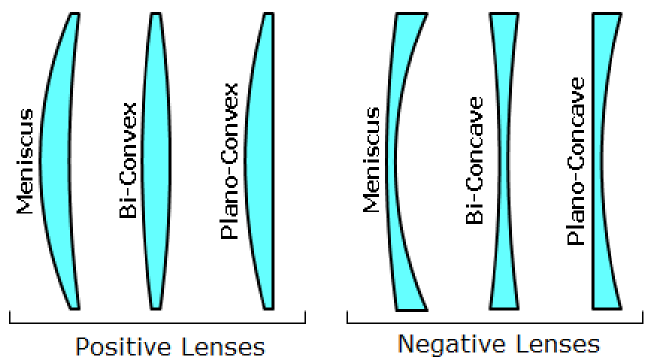
$$\mathcal{R} = \begin{bmatrix} 1 & -D \\ 0 & 1 \end{bmatrix};\; D = \frac{n_t - n_i}{R};\; \mathcal{T} = \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix};\; \mathscr{r} = \begin{bmatrix} n\alpha \\ y \end{bmatrix};\; \mathcal{A}_{n1} = \mathcal{R}_n\, \mathcal{T}_{n,n\text{-}1} \mathcal{R}_{n-1} \ldots \mathcal{R}_1.$$

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f_o} = \frac{1}{f_i} = \frac{1}{f} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right);\; \mathcal{O} = \begin{bmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{bmatrix}$$

For $y_i \neq y_i(\alpha_o), \mathcal{O}_{21} = 0$!!

Some types of lenses and sign conventions.

Lenses are broadly characterized as converging (positive, convex) or diverging (negative, concave). Lenses may also be characterized by their shape, i.e., planar (plano) – convex, biconvex, positive meniscus, planar (plano) – concave, biconcave, or negative meniscus. As a general rule, lenses that are thicker in the middle than at the edges are converging lenses, and lenses that are thinner in the middle that at the edges are diverging. The sign convention is as follows for lenses.



Categories of Lens Form

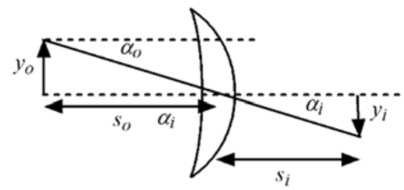
$$\begin{gathered} s_o > 0 \Rightarrow \text{real object};\; s_o < 0 \Rightarrow \text{virtual object} \\ s_i > 0 \Rightarrow \text{real image};\; s_i < 0 \Rightarrow \text{virtual image} \\ f > 0 \Rightarrow \text{converging lens};\; f < 0 \Rightarrow \text{diverging lens} \\ y_o > 0 \Rightarrow \text{upright object};\; y_o < 0 \Rightarrow \text{inverted object} \\ y_i > 0 \Rightarrow \text{upright image};\; y_i < 0 \Rightarrow \text{inverted image} \\ R > 0 \Rightarrow \text{C is right of surface};\; R_i < 0 \Rightarrow \text{C is to the left of surface} \end{gathered}$$

Most books assume that light enters the optical system from the left-hand side and exits to the right-hand side. This means that $s_o$is usually taken to be positive when the object is to the left of the lens, and that s_i is positive when the image is to the right of the lens. Most of the time, the location of the object and image will result in an image that is either enlarged or reduced in size. We introduce the lateral (or transverse) magnification M_T to describe the image size. The definition of M_T is given by

$$M_T = \frac{y_i}{y_o} = -\frac{s_i}{s_o}.$$

It is not difficult to prove that when multiple lenses are involved, magnifications are multiplied. Here is a numerical example using a thin lens.

Example: The lens shown to the right has the following parameters that describe it. Determine the image position, and the magnification of the lens.



$$|R_1| = 10 \text{ cm}; \ |R_2| = 5 \text{ cm}; n_l = \frac{3}{2}$$

$$s_o = 30 \text{ cm}$$

$$\frac{1}{f} = (n_l - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \left(\frac{3}{2} - 1\right)\left(\frac{1}{-10} - \frac{1}{-5}\right) \Rightarrow f = 20 \text{ cm}$$

$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i} \Rightarrow \frac{1}{s_i} = \frac{1}{f} - \frac{1}{s_o} = \frac{1}{20} - \frac{1}{30} = \frac{3}{60} - \frac{2}{60} \Rightarrow s_i = 60 \text{ cm}$$

Therefore, the magnification is given by $M_T = -\frac{s_i}{s_o} = -\frac{60}{30} = -2$.
This means the image is twice the size of the object and is inverted, so our drawing is not to scale.

Let's see what happens when we insert a second lens in this system. Let's put our second lens 40 cm to the right of the first lens and give it a focal length of 10 cm. To treat the case as you would have in your introductory course, the image of the first lens acts as the object for our second lens. This lens, however, is in front (to the left) of the image of the first lens. Therefore, light is converging onto this lens, and the object is said to be virtual. If our second lens were placed at 80 cm to the right of the first lens, then light from the image of the first lens would be diverging onto the lens, and the object would be real. Determining whether light converges or diverges onto a lens determines whether the object is virtual or real. Now let's put the numbers in.

$$\frac{1}{f_2} = \frac{1}{s_{o2}} + \frac{1}{s_{i2}} \Rightarrow \frac{1}{s_{i2}} = \frac{1}{f_2} - \frac{1}{s_{o2}} = \frac{1}{10} - \frac{1}{-20} = \frac{1}{10} + \frac{1}{20} = \frac{3}{20} \Rightarrow s_{i2} = \frac{20}{3} = 6.67 \text{ cm}$$

The magnification for this second lens is given by

$M_{T2} = -\frac{s_{i2}}{s_{o2}} = -\frac{20/3}{-20} = \frac{1}{3}$ and the system magnification is $(-2)\left(\frac{1}{3}\right) = -\frac{2}{3}$.

The final image is now 6.67 cm to the right of the second lens, which means that it is now only 46.67 cm to the right of the first lens. The short focal length of the second lens has drawn the image closer to the original lens and reduced it in size, but it is still inverted.

Let's go ahead and derive the magnification in terms of the image and object distances.

$$u_i = \begin{bmatrix} \alpha_i \\ y_i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s_i & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s_o & 1 \end{bmatrix} \begin{bmatrix} \alpha_o \\ y_o \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{f} \\ s_i & 1 - \frac{s_i}{f} \end{bmatrix} \begin{bmatrix} \alpha_o \\ s_o \alpha_o + y_o \end{bmatrix}.$$

Now notice that $\alpha_o = -\frac{y_o}{s_o}$, so $\begin{bmatrix}\alpha_i\\ y_i\end{bmatrix} = \begin{bmatrix}1 & -\frac{1}{f}\\ s_i & 1-\frac{s_i}{f}\end{bmatrix}\begin{bmatrix}\alpha_o\\ 0\end{bmatrix} = \begin{bmatrix}\alpha_o\\ \alpha_o s_i\end{bmatrix}$ and $y_i = \alpha_o s_i = -\frac{y_o}{s_o} s_i$.

Finally,

$$\frac{y_i}{y_o} = M_T = -\frac{s_i}{s_o}.$$

Let's consider what happens in a formal way when we have multiple thin lenses to deal with. In your introductory class, you dealt with multiple thin lenses by finding the image of the object, treating that image as the object for the second lens, then treating its image as the object for the next lens, and so forth. While this is a perfectly valid method of dealing with a multiple lens system, it is not the most efficient way, particularly if you have to make many changes in the object position and determine the final image position and characteristics many times. We now reap the benefits of the matrix formulation of optical imaging. We may calculate the system matrix one time for any complicated arrangement of lenses and then use the object-image matrix to determine the correct relationship to use to relate the object and image distances. We will keep our lens in air for now and use only two lenses to generalize our procedure. Let s_{o1} be the object location and s_{i2} be the image location. The object and image ray vectors are given by

$$\boldsymbol{r}_o = \begin{bmatrix}\alpha_o\\ y_o\end{bmatrix} \text{ and } \boldsymbol{r}_i = \begin{bmatrix}\alpha_i\\ y_i\end{bmatrix}.$$

Let me write the full matrix equation and then I will go over what everything means.

$$\begin{bmatrix}\alpha_i\\ y_i\end{bmatrix} = \begin{bmatrix}1 & 0\\ s_{i2} & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{f_2}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ d & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{f_1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ s_{o1} & 1\end{bmatrix}\begin{bmatrix}\alpha_o\\ y_o\end{bmatrix}$$

Remember we work from right to left to interpret the equation.

$$\begin{bmatrix}\alpha_i\\ y_i\end{bmatrix} = \begin{bmatrix}1 & -\frac{1}{f_2}\\ s_{i2} & 1-\frac{s_{i2}}{f_2}\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{f_1}\\ d & 1-\frac{d}{f_1}\end{bmatrix}\begin{bmatrix}1 & 0\\ s_{o1} & 1\end{bmatrix}\begin{bmatrix}\alpha_o\\ y_o\end{bmatrix}$$

$$= \begin{bmatrix}1 & -\frac{1}{f_2}\\ s_{i2} & 1-\frac{s_{i2}}{f_2}\end{bmatrix}\begin{bmatrix}1-\frac{s_{o1}}{f_1} & -\frac{1}{f_1}\\ d+s_{o1}-\frac{s_{o1}d}{f_1} & 1-\frac{d}{f_1}\end{bmatrix}\begin{bmatrix}\alpha_o\\ y_o\end{bmatrix}$$

$$= \begin{bmatrix}1-\frac{s_{o1}}{f_1}-\frac{1}{f_2}\left(d+s_{o1}-\frac{s_{o1}}{f_1}\right) & -\frac{1}{f_1}-\frac{1}{f_2}\left(1-\frac{d}{f_1}\right)\\ s_{i2}\left(1-\frac{s_{o1}}{f_1}\right)+\left(1-\frac{s_{i2}}{f_2}\right)\left(d+s_{o1}-\frac{s_{o1}d}{f_1}\right) & -\frac{s_{i2}}{f_1}+\left(1-\frac{s_{i2}}{f_1}\right)\left(1-\frac{d}{f_1}\right)\end{bmatrix}\begin{bmatrix}\alpha_o\\ y_o\end{bmatrix}$$

Now we apply the imaging condition, $\mathcal{O}_{21} = 0$ and simplify.

$$s_{i2}\left(1 - \frac{s_{o1}}{f_1}\right) + \left(1 - \frac{s_{i2}}{f_2}\right)\left(d + s_{o1} - \frac{s_{o1}d}{f_1}\right) = 0$$

Rewriting gives

$$s_{i2}\left[\left(\left(1 - \frac{s_{o1}}{f_1}\right)\right) - \frac{1}{f_2}\left(\left(d + s_{o1} - \frac{s_{o1}d}{f_1}\right)\right)\right] = -d - s_{01}\left(1 - \frac{d}{f_1}\right)$$

and

$$s_{i2} = \frac{s_{o1}\left(\frac{d}{f_1} - 1\right) - d}{1 - \frac{s_{o1}}{f_1} - \frac{1}{f_2}\left(d + s_{o1} - \frac{s_{o1}d}{f_1}\right)}.$$

We may put this in the form of your textbook (Eq. 5.33) to obtain

$$s_{i2} = \frac{f_2 d - f_2 s_{o1} f_1/(s_{o1} - f_1)}{d - f_2 - s_{o1} f_1/(s_{o1} - f_1)}.$$

We achieved our goal of getting a relationship between the object and image distances, but it is not pretty.

Three points are important about our process and this relationship.

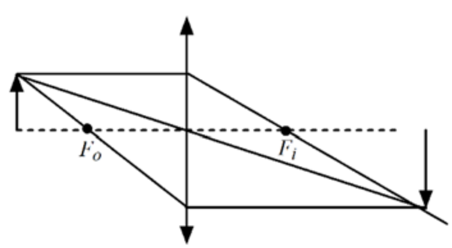
1. We may always use $\mathcal{O}_{21} = 0$ as the imaging condition.
2. The relationship between s_{i2} and s_{o1} is not simple.
3. The front and back focal lengths are not the same even in air.

$$\lim_{s_{o1}\to\infty} s_{i2} = bfl = \frac{f_2(d - f_1)}{d - (f_1 + f_2)}$$

and

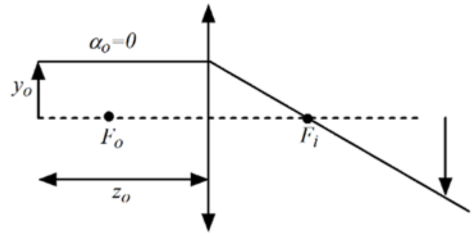
$$\lim_{s_{i2}\to\infty} s_{o1} = ffl = \frac{f_1(d - f_2)}{d - (f_1 + f_2)}.$$

Let's take a look at how our matrix approach predicts the standard ray tracing you learned in your introductory course. You might recall that you are usually able to trace 3 rays (called principal rays) through a lens. Here is a figure showing how it works for a thin lens. Does this make good sense using our matrix approach to determine how the rays should behave? Let's just look to see how the input parallel ray behaves. Our object ray vector in air is given by



$$\begin{bmatrix}\alpha_o \\ y_o\end{bmatrix} = \begin{bmatrix}0 \\ y_o\end{bmatrix}.$$

$$\begin{bmatrix}\alpha_i\\ y_i\end{bmatrix}=\begin{bmatrix}1 & 0\\ z_i & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{f}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ z_o & 1\end{bmatrix}\begin{bmatrix}0\\ y_o\end{bmatrix}$$



$$=\begin{bmatrix}1 & 0\\ z_i & 1\end{bmatrix}\begin{bmatrix}1-\frac{z_o}{f} & -\frac{1}{f}\\ z_o & 1\end{bmatrix}\begin{bmatrix}0\\ y_o\end{bmatrix}$$

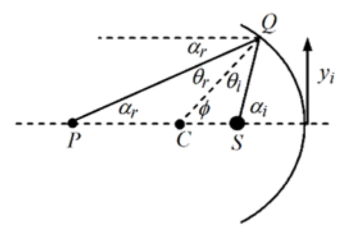
$$=\begin{bmatrix}1-\frac{z_o}{f} & -\frac{1}{f}\\ z_i\left(1-\frac{z_o}{f}\right)+z_o & -\frac{z_i}{f}+1\end{bmatrix}\begin{bmatrix}0\\ y_o\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{y_o}{f}\\ \left(-\frac{z_i}{f}+1\right)y_o\end{bmatrix}=\begin{bmatrix}\alpha_i\\ y_i\end{bmatrix}$$

Therefore. we obtain $\alpha_i=-\frac{y_o}{f}$ and $y_i=\left(-\frac{z_i}{f}+1\right)y_o$. We wish to see where the refracted ray from the lens crosses the optical axis so we want $y_i=0$. Finally, $-\frac{z_i}{f}+1=0$ and $z_i=f$. This means that a ray parallel to the optical axis crosses the optical axis at the focal point of the lens. We could do the same thing for the other two principal rays, but the approach is quite similar, so I will not repeat the calculation here.

We have discussed thin lenses and thin lens combinations, but what about mirrors? They have their limitations, but they can be quite useful. Do you see what one of their major limitations is and what one of their nice advantages is?

We may treat mirrors in the same way we treated lenses using the small angle approximation. Here is a figure that shows the relevant parameters to describe a mirror. Note here that the angle are not small, but that is just for convenience of drawing the figure.



$$\alpha_i=\phi+\theta_i \Rightarrow \alpha_i-\theta_i=-\frac{y_i}{R} \text{ since } R<0.$$

$$\phi=\alpha_r+\theta_r \Rightarrow \alpha_r=\phi-\theta_r=\phi-\theta_i=-\frac{y_i}{R}-\frac{y_i}{R}-\alpha_i=-\alpha_i-2\frac{y_i}{R}$$

$$n\alpha_r=-n\alpha_i-2\frac{ny_i}{R} \text{ and } y_r=y_i$$

Finally,

$$\begin{bmatrix}n\alpha_r\\ y_r\end{bmatrix}=\begin{bmatrix}-1 & -\frac{2n}{R}\\ 0 & 1\end{bmatrix}\begin{bmatrix}n\alpha_i\\ y_i\end{bmatrix}.$$

To see what the imaging condition for mirrors is, we may use the same approach as we did for lenses. We put n = 1.

$$\begin{bmatrix}\alpha_r\\ y_r\end{bmatrix} = \begin{bmatrix}1 & 0\\ -s_i & 1\end{bmatrix}\begin{bmatrix}-1 & -\frac{2}{R}\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ s_o & 1\end{bmatrix}\begin{bmatrix}\alpha_i\\ y_i\end{bmatrix}$$

$$\begin{bmatrix}\alpha_r\\ y_r\end{bmatrix} = \begin{bmatrix}\left(-1-\frac{2s_o}{R}\right) & -\frac{2}{R}\\ s_i + s_o\left(1+\frac{2s_i}{R}\right) & \left(1-\frac{2s_i}{R}\right)\end{bmatrix}\begin{bmatrix}\alpha_i\\ y_i\end{bmatrix}$$

Now we use $\mathcal{O}_{21} = 0$ again.

$$s_i + s_o\left(1+\frac{2s_i}{R}\right) = 0$$

and

$$s_i + s_o + \frac{2s_o s_i}{R} = 0.$$

Therefore,

$$\frac{1}{s_o} + \frac{1}{s_i} = -\frac{2}{R} = \frac{1}{f}$$

since

$$\lim_{s_i\to\infty} s_o = f_o \text{ and } \lim_{s_o\to\infty} s_i = f_i.$$

Concave mirrors are converging elements ($R < 0$, $f > 0$), and convex mirrors are diverging elements with $R < 0$, $f > 0$.

NEXT TIME: Prisms, optical fibers, and thick lenses