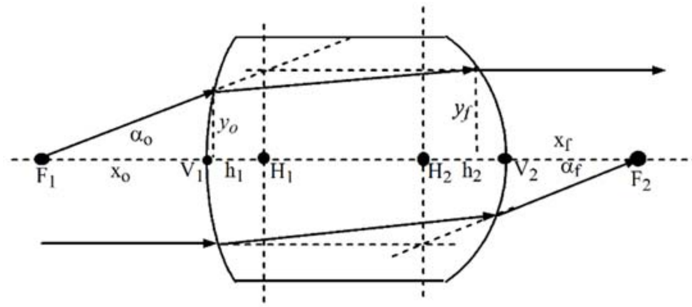
LAST TIME: Prisms, fiber optics, and introduction to thick lenses, and principal points

Recall

$V_1H_1 = h_1$ $F_1H_1 = f_1$ $V_2H_2 = h_2$
$F_2H_2 = f_2$ $x_o = F_1V_1$ $x_f = F_2V_2$.

Therefore, $f_1 = x_o + h_1$ and $f_2 = x_f + h_2$.

Finally, $f_1 = -\frac{a_{11}a_{22} - a_{21}a_{12}}{a_{12}} = -\frac{1}{a_{12}}$



$$h_1 = f_1 - x_o = -\frac{1}{a_{12}} - \frac{y_o}{\alpha_o} = -\frac{1}{a_{12}} - -\frac{a_{11}}{a_{12}} = \frac{1 - a_{11}}{-a_{12}}.$$

Similarly,

$$h_2 = \frac{a_{22} - 1}{-a_{12}}$$

What does all this mean?

We have established that the points H_1 and H_2 are points where refraction can be thought of to bo concentrated. We have also establishe that the distances h_1 and h_2 are simply related to the system matrix. We still do not have a simply relationship between s_o, s_i, and f. Let's see what happens when so and si are measured with respect to H_1 and H_2. We will trace a ray from an object to its image and use the imaging condition to determine the relationship. I will write the matrix operations first and then explain what is going on.

$$\begin{bmatrix}\alpha_i \\ y_i\end{bmatrix} = \begin{bmatrix}1 & 0 \\ s_i & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ h_2 & 1\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1 & 0 \\ -h_1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ s_o & 1\end{bmatrix}\begin{bmatrix}\alpha_o \\ y_o\end{bmatrix}$$

Remember to read the process from right to left to get the right order through the optical system. Here is the result of the entire matrix mutiplication.

$$\begin{bmatrix}\alpha_i \\ y_i\end{bmatrix} = \begin{bmatrix} a_{12}s_o + 1 & a_{12} \\ \frac{[a_{11}a_{22} - a_{12}^2 s_i s_o - a_{12}(a_{21} + s_i + s_o) - 1]}{a_{12}} & a_{12}s_i + 1\end{bmatrix}\begin{bmatrix}\alpha_o \\ y_o\end{bmatrix}$$

Now we set $\mathcal{O}_{21} = 0$ for the imaging condition to obtain

$$\frac{[a_{11}a_{22} - a_{12}^2 s_i s_o - a_{12}(a_{21} + s_i + s_o) - 1]}{a_{12}} = 0$$

This gives $[a_{11}a_{22} - a_{12}^2 s_i s_o - a_{12}(a_{21} + s_i + s_o) - 1] = 0$,

so $1 - a_{12}^2 s_i s_o - a_{12}s_i - a_{12}s_o - 1 = 0$

and $s_i + s_o = -a_{12}s_i s_o \Longrightarrow \frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$

Let's summarize our results.

For any optical system, there exist 2 points, H1 and H2 (principal points) at which refraction may be considered to be concenrated. When the system is in air,

$$h_1 = \frac{1 - a_{11}}{-a_{12}}, \qquad h_2 = \frac{a_{22} - 1}{-a_{12}}, \qquad f = -\frac{1}{a_{12}} \text{ and } \frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f},$$

where object and image distances, and focal lengths are measured with respect to H_1 and H_2. It is also possible to find H_1 and H_2 for each element in the system and apply the thin lens equation *n* times. It is also possible to apply the object-image matrix without regard to H_1 and H_2, but in this case, object and image distances are measured from V_{11} and V_{2n}.

For a single thick lens the result are given by

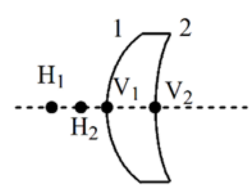
$$h_1 = -\frac{f(n_l - 1)d_l}{R_2 n_l}, h_2 = -\frac{f(n_l - 1)d_l}{R_1 n_l} \text{ and } \frac{1}{f} = (n_l - 1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_l - 1)d_l}{n_l R_1 R_2}\right].$$

To derive the last equation, we just write the system matrix for a thick lens and identify

$$f = -\frac{1}{a_{12}}.$$

Here is an example. Suppose we have a thick lens with the following properties.

$$|R_1| = 6.00 \text{ cm}, |R_2| = 10.0 \text{ cm}, d_l = 3.00 \text{ cm, and } n_l = \frac{3}{2}.$$



We notice here that both centers of curvature of the surfaces are positive because they are to the right of the surface. We will locate an object 38.4 cm from V_1, and we wish to characterize the image. First, calculate the focal length using

$$\frac{1}{f} = (n_l - 1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n_l - 1)d_l}{n_l R_1 R_2}\right] = \frac{1}{2}\left[\frac{1}{6} - \frac{1}{10} + \frac{\frac{1}{2}(3)}{\frac{3}{2}(6)(10)}\right] = \frac{5}{120}$$

and $f = 24.0$ cm. Now we find h_1 and h_2.

$$h_1 = -\frac{f(n_l - 1)d_l}{R_2 n_l} = -\frac{24\left(\frac{1}{2}\right)3}{10\left(\frac{3}{2}\right)} = -\frac{36}{15} = -2.40 \text{ cm}$$

$$h_2 = -\frac{f(n_l - 1)d_l}{R_1 n_l} = -\frac{24\left(\frac{1}{2}\right)3}{6\left(\frac{3}{2}\right)} = -4.00$$

Our object distance is measured with respect to H_1, so it is 38.4+h_1 = 38.4 – 2.4 = 36.0 cm. We apply the thin lens equation to obtain

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f} \Longrightarrow \frac{1}{s_i} = \frac{1}{f} - \frac{1}{s_o} = \frac{1}{24} - \frac{1}{36} = \frac{3}{72} - \frac{2}{72} \Longrightarrow s_i = 72.0 \text{ cm from } H_2.$$

To find its location relative to V_2, we get 72 – 4 = 68 cm from V_2. Therefore, the image magnification is given by

$$M_T = -\frac{72}{36} = -2.00.$$

The image is real, inverted, and magnified by a factor of 2.

Here is another example using the same thick lens as above. Suppose a ray 14.4 cm from V1 starts on the axis and makes an angle of 0.2 rad with respect to the optical axis. Determine the location at which the ray crosses the optical axis. Find the system matrix first. Use $D = \frac{n_t - n_i}{R}$ and obtain

$$D_1 = \frac{n_t - n_i}{R_1} = \frac{n_l - 1}{R_1} = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{12}\text{cm}^{-1} \; and \; D_2 = \frac{1 - n_l}{R_2} = -\frac{1}{20}\text{cm}^{-1}$$

Therefore,

$$\boldsymbol{a}_{21} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{20} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{3}{\frac{3}{2}} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{10} & -\frac{1}{24} \\ 2 & \frac{5}{6} \end{bmatrix}.$$

Check to be sure the determinant is one, so $\frac{55}{60} + \frac{5}{60} = 1$ as required.
Now create our matrix to get our ray through.

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} \frac{11}{10} & -\frac{1}{24} \\ 2 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 14.4 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{10} & -\frac{1}{24} \\ \left(\frac{11s}{10}\right)2 & -\frac{s}{24} + \frac{5}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{5} \\ \frac{72}{25} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{50} - \frac{1}{24}\left(\frac{72}{25}\right) \\ \left(\frac{11s}{50}\right) + \frac{2}{5} - \frac{s}{24}\left(\frac{72}{25}\right) + \frac{5}{6}\left(\frac{72}{25}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{50} - \frac{3}{25} \\ \left(\frac{11s}{50}\right) - \frac{3s}{25} + \frac{2}{5} + \frac{12}{5} \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}.$$

Finally, $\alpha = 0.1$ and $\frac{s}{10} = -\frac{28}{10}$ so $s = -28$ cm

NEXT TIME: Another example, lens aberrations