LAST TIME: Reflectance, transmittance, polarization states, linear polarizers, and anisotropic materials

$$R = \left(\frac{E_{or}}{E_{oi}}\right)^2 = r^2; \quad T = \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\left(\frac{E_{ot}}{E_{oi}}\right)^2 = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} t^2 \quad R + T = 1; \quad R + T + A = 1$$

$$\left(\frac{E_y}{E_{oy}}\right)^2 - 2\frac{E_x E_y}{E_{ox}E_{oy}}\cos\epsilon + \left(\frac{E_x}{E_{ox}}\right)^2 =; \quad 2\alpha = \tan^{-1}\frac{2E_{ox}E_{oy}\cos\epsilon}{E_{ox}^2 - E_{oy}^2}$$

$$\frac{OPD}{\lambda} = \frac{\pi/2}{2\pi} = \frac{1}{4} \text{ so } OPD = \frac{\lambda}{4} = \left|n_f - n_s\right| d.$$

Linear polarizers

The most common type of linear polarizers is an H sheet. The designation is to use the term HN-xx. The H tells us the type of polarize, the N tells us that its color is neutral, and the xx gives the percentage of light transmitted if the incident light is randomly polarized. An HN-50 is an ideal polarizer because it transmits 50% of the incident randomly polarized light incident on it. Furthermore, it transmits 100% of the incident polarized light that it parallel to the transmission axis. The reason for this is that randomly polarized light has 50% perpendicular to the transmission axis and 50% parallel to the transmission axis.

Example: Randomly polarized light having irradiance I_i is incident on two HN-32 polarizers whose TAs are parallel. Determine the irradiance of the light that emerges from the system.

After LP1, $I_{LP1} = 0.32 I_i$. When it goes into LP2, it is polarized, so it transmits 0.64 of the light incident on it. Therefore, $I_{LP2} = 0.64 I_{LP1} = 0.64(0.32 I_i) = 0.2048 I_i$. Now suppose that the transmission axis of LP2 makes an angle of 30 degrees with respect to the transmission axis of LP1. What is the transmitted irradiance? $I_{LP2} = 0.64 I_{LP1}\cos^2 30^o = 0.64(0.32 I_i)\cos^2 30^o$. Therefore, $I_{LP2} = 0.1536 I_i$.

To understand the problem of anisotropic materials, we need to go back to another system that has some mathematical similarities. Recall that when we had arbitrary rotation about an axis, we had to describe the situation using some thing called a moment of inertia tensor. We wrote $L_i = I_{ij}\omega_j$. Anisotropic materials also connect indices of refraction and dielectric coefficients to the relevant fields such as **D** and **E**. These complications result in some rather difficult equations to deal with, so I will show you some key results from another book. The figures and equations result from solving the wave equation when the connections between **D** and **E** are given by a tensor. Another important connection between optical constants and the electric fields is through the electric susceptibility χ_e. The relationship between the electric polaization P and the electric field E is given by

$$\mathbf{P} = \epsilon_o \chi \mathbf{E}$$

for isoptropic materials. However, for anisotropic materials, we would write

$$P_i = \epsilon_o \chi_{ij} E_j.$$

Remind folks about the moment of inertia tensor and its connection to angular velocity and angular momentum. The relationship between the electric displacement vector **D** and the electric field **E** is given by

$$D_i = \epsilon_o \left(\delta_{ij} + \chi_{ij}\right) E_j.$$

The wave equation with these relationships is more complicated than what we wrote earlier because of the possiblity that the vectors **P**, **E**, and **D** are no longer parallel. The wave equation becomes

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{1}{c^2}\overleftrightarrow{\boldsymbol{\chi}}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

We assume that we have a plane wave propagating in the material so the curls become cross products and the time derivatives become $-\omega^2$, and we obtain

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \left(\frac{\omega}{c}\right)^2 = -\left(\frac{\omega}{c}\right)^2 \overleftrightarrow{\boldsymbol{\chi}}\mathbf{E}.$$

We understand now that we are working in the principal axis coordinate system so we have only three values of χ_{ij} to worry about. Three equations follow from this treatment given by

$$\left(-k_y^2 - k_z^2 + \frac{\omega^2}{c^2}\right) E_x + k_x k_y\, E_y + k_x k_z\, E_z = -\frac{\omega^2}{c^2}\,\chi_{11}\, E_x$$

$$k_y k_x\, E_x + \left(-k_x^2 - k_z^2 + \frac{\omega^2}{c^2}\right) E_y + k_y k_z\, E_z = -\frac{\omega^2}{c^2}\,\chi_{22}\, E_y$$

$$k_z k_x\, E_x + k_z \mathrm{k}_y\, E_y + \left(-\,k_x^2 - k_y^2 + \frac{\omega^2}{c^2}\right) E_z = -\frac{\omega^2}{c^2}\,\chi_{33}\, E_z$$

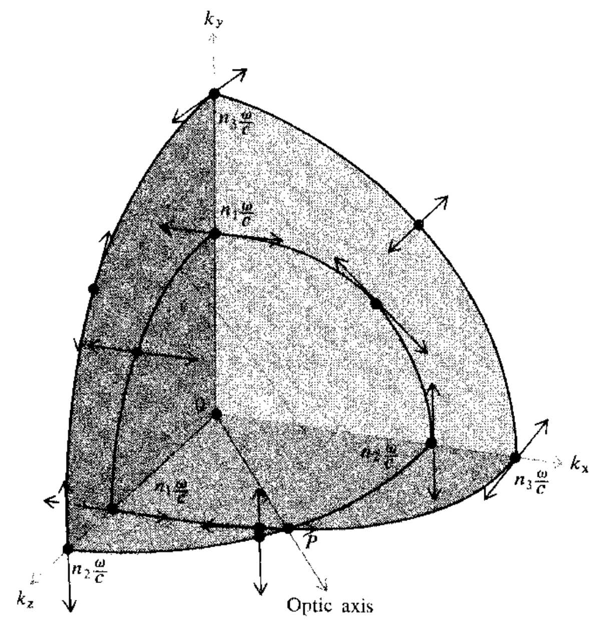
The principal indices of refraction are related to the electric susceptibility using equations given by

$$n_1 = \sqrt{1 + \chi_{11}} = \sqrt{K_{11}}$$
$$n_2 = \sqrt{1 + \chi_{22}} = \sqrt{K_{22}}$$
$$n_3 = \sqrt{1 + \chi_{33}} = \sqrt{K_{33}}$$

For a nontrivial solution for Ex, Ey, and Ez to exist, the determinant of the coefficients must vanish. Therefore, we obtain

$$\begin{vmatrix} (n_1\omega/c)^2 - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & (n_2\omega/c)^2 - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & (n_3\omega/c)^2 - k_x^2 - k_y^2 \end{vmatrix} = 0$$

This equation represents a rather complicated 3-dimensional surface in k space. It is called the wave-vector surface. It is shown on the next page.



The only way that makes much sense to interpret this equation is to consider each of the coordinate planes separately. In the plane $k_z = 0$, the determinant reduces to

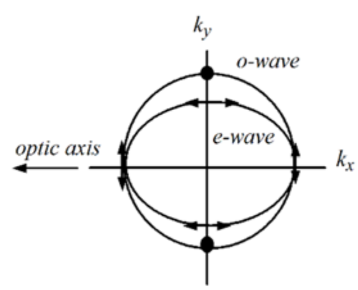
$$\left[\left(\frac{n_3\omega}{c}\right)^2 - k_x{}^2 - k_y{}^2\right]\left\{\left[\left(\frac{n_1\omega}{c}\right)^2 - k_y{}^2\right]\left[\left(\frac{n_2\omega}{c}\right)^2 - k_x{}^2\right] - k_x{}^2k_y{}^2\right\} = 0$$

If we set the first part of the equation equal to zero, we see that we get the equation of a circle given by

$$k_x{}^2 + k_y{}^2 = \left(\frac{n_3\omega}{c}\right)^2$$

If we set the second part equal to zero, we obtain

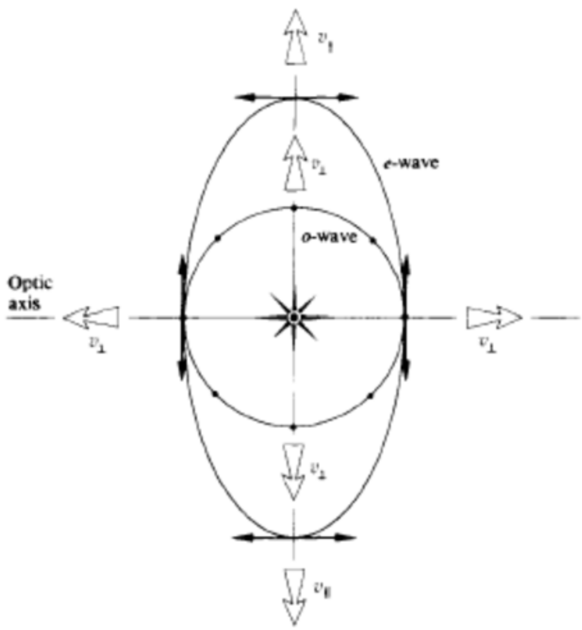
$$\frac{k_x{}^2}{(n_2\omega/c)^2} + \frac{k_y{}^2}{(n_1\omega/c)^2} = 1$$

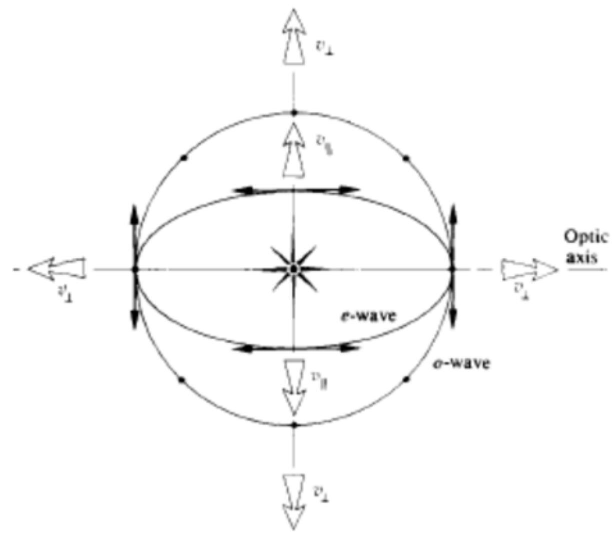


which is the equation of an ellipse. In order to make more progress, we need to know something about the relative values of the indices of refraction. Crystals typically fall into one of three categories: isotropic, uniaixal, and biaxial. For isotropic materials, all values of the refractive indices are equal, for uniaxial crystals, two refractive indices are equal, and for biaxial crystals, all three indices of refraction are different. For the case considered above, let's suppose $n_2 = n_3$ and $n_1 < n_3$. Then, the our figure then appears as shown. The outside circle has as radius of $n_3\,\frac{\omega}{c}$ and the ellipse has a major axis of $n_3\,\frac{\omega}{c}$ and a mionor axis of $n_1\,\frac{\omega}{c}$. Along k_x, the two state of polarization have the same speed, by along k_y, they are different. The birefringence is given by $\Delta n = n_e - n_o$. If the difference is negative, we say the crystal is negative uniaxial and n_e is less than n_o, but if the difference is positive, we say that the crystal is uniaxial is positive and n_e is greater than n_o. Mathematically, we see the following.

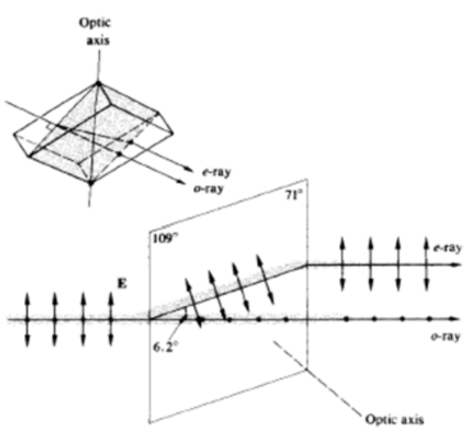
$$n_e < n_o \Longrightarrow v_e > v_o \Longrightarrow v_\parallel > v_\perp \quad \text{and } n_e > n_o \Longrightarrow v_e < v_o \Longrightarrow v_\parallel < v_\perp$$

For the ordinary wave, the electric field is perpendicular to the optic axis everywhere, but for the extraordinary wave, this is not true. Here are similar figures from your textbook.





Waves that travel in these materials are very much different from those that travel in isotropic materials. Here are some figures that show how the electric field, electric displacement field, Poynting vector, and **k**-vector look in such cases. Double refraction can be understood using the following figures.



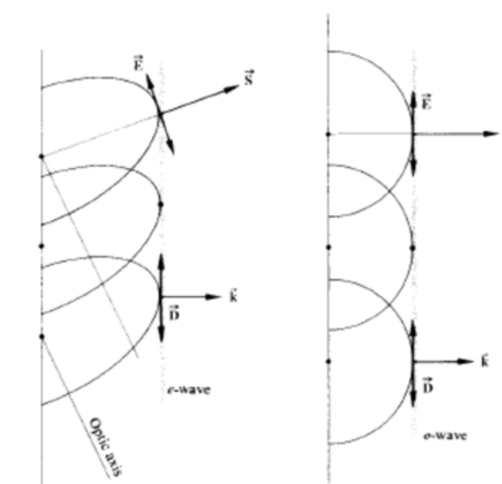
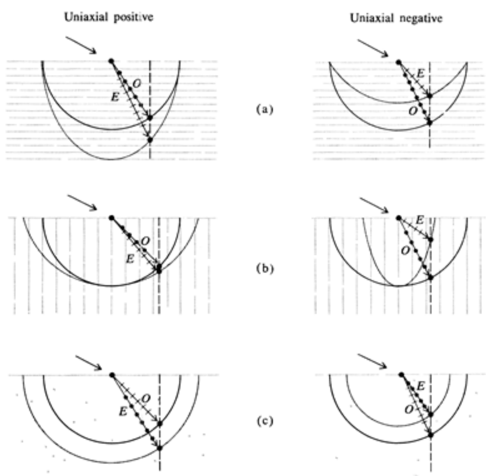
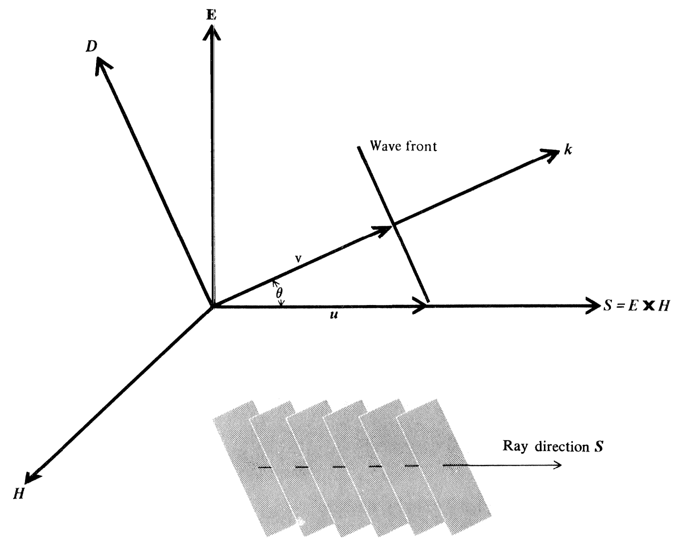


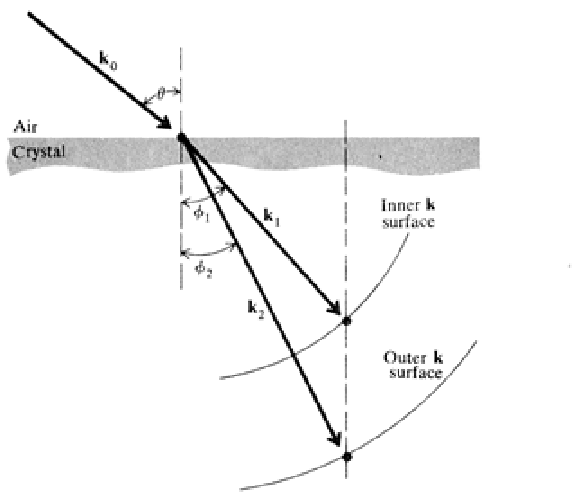
Figure 8.23 Orientations of the $\vec{E}$-, $\vec{D}$-, $\vec{S}$-, and $\vec{k}$-vectors.

Here are some other figures showing some of these strange behaviors.



Wave vectors for double refraction in uniaxial crystals. (a) The optic axis parallel to the boundary and parallel to the plane of incidence; (b) the optic axis perpendicular to the boundary and parallel to the plane of incidence; (c) the optic axis parallel to the boundary and perpendicular to the plane of incidence.





Wave vectors for double refraction at the boundary of a crystal.

we can write

$$k_0 \sin\theta = k_1 \sin\phi_1 \qquad k_0 \sin\theta = k_2 \sin\phi_2$$

for the two refracted waves.

As you can see, the problem of crystal optics is considerably more complicated than that of isotropic materials. I have only tried to give you a flavor of how things work, and I do not expect that all of you will get the same thing from this discussion. We can, however, give some names to the discussion we had on Monday to the fast and slow axes. If we have a crystal with the optic axis parallel to its surface, there will be a difference between the two states of linearly polarized light. This accumulates our phase difference, so $OPD = d|n_e - n_o|\ and$ our phase difference is once again given by

$$\epsilon = \frac{2\pi}{\lambda} OPD,$$

So as we saw before, we can get our requirement to get a $\pi/2$ phase difference between the two waves. Therefore, the thickness of our plate must be given by

$$d = \frac{\lambda_o}{4|n_e - n_o|}.$$

There is another very useful way to give a mathematical description of polarization called the Jones vectors and the Jones matrices. Here is how it works. We know that we can represent a plane wave using the exponential notation $E_x = E_{ox}\exp(kz - \omega t)$ and $E_y = E_{oy}\exp(kz - \omega t + \epsilon)$. Evidently, only three parameters are required to specify the SOP (state of polarization). Therefore, we might try to write the polarization vector as a column vector given by

$$\begin{bmatrix} E_{ox} \\ E_{oy}e^{i\epsilon} \end{bmatrix}.$$

We just need to specify each of the parameters to get the SOP. These are know as the Jones vectors, and there are a few to represent some states of polarization. We will determine what the states are in class.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

You will need to recall that $e^{i\epsilon} = \cos\epsilon + i\sin\epsilon$. In the example above, $e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$. Writing these Jones vectors would not be of much use if we did not have some way to represent the action of linear polarizers and wave plates. We must find ways to represent these objects. The only way to do this is to let a particular polarizing element act on a state of polarization and equate it to wha t we know its effect should be. Here is an example of how to do this for one case. Suppose we wish to know how a horizontal linear polarizer is represented.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \Longrightarrow a = 1$ and $c = 0$. Similarly, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow b = 0$ and $d = 0$.

Therefore, a Jones matrix that represents a horizontal linear polarizer is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The Jones matrices are found ini exactly the same way. We may have as many as four unknowns, so we will need that many equations.

NEXT TIME: Examples using different states of polarization with and without Jones vectors and Jones matrices