LAST TIME: Two-wave interference and examples

We begin by adding two waves having the same frequency with a relative phase of ϵ. The two waves are given by

$$\mathbf{E}_1 = \mathbf{E}_{o1} e^{i(kr_1 - \omega t)}$$

and

$$\mathbf{E}_2 = \mathbf{E}_{o2} e^{i(kr_2 - \omega t + \epsilon)}.$$

$$I = \frac{1}{2} Re(\mathbf{E} \cdot \mathbf{E}^*), \text{which comes from } \langle \text{fg} \rangle = \frac{1}{2} Re(f g^*)$$
$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos[k(r_1 - r_2) + \epsilon].$$

The total phase difference breaks up neatly into two components, the phase difference due to the optical path difference $[k(r_1 - r_2)]$ and the remaining phase ϵ, which comes from the inherent phase difference + the phase differences due to reflections. I believe this is the most effective way to consider phase in interference problems. Usually, in optics, an inherent phase difference is difficult to build in, but in electronically controlled devices, it is relatively easy to do. We call the total phase difference $\delta = \delta_{OPD} + \delta_{in} + \delta_{ref}$. We note the following:

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}; \;\; \delta = 0, \pm 2\pi, \cdots \pm 2n\pi$$

and

$$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}; \;\; \delta = \pm\pi, \cdots \pm (2n+1)\pi$$

The fringe visibility is defined to be

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2},$$

so

$$V = 1$$

if

$$I_1 = I_2 = I_o.$$

Young's interference gives

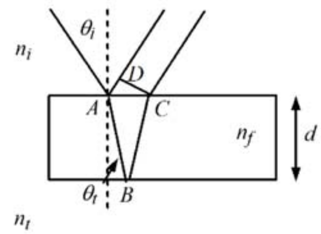
$$\delta = k(r_1 - r_2) = \frac{2\pi}{\lambda} OPD,$$

so $I(y) = 4I_o \cos \frac{2\pi}{\lambda}\frac{ay}{2s} = 4I_o \cos^2\left(\frac{ya\pi}{\lambda s}\right)$.

The maxima occur whenever the argument of the cosine function is given by $m\pi$. Therefore,

$\frac{y_m a\pi}{\lambda s} = m\pi$ which implies that $y_m = \frac{m\lambda s}{a}$. The constructive interference fringes occur at equally spaced intervals given by $y_{m+1} - y_m = \frac{(m+1)\lambda s}{a} - \frac{m\lambda s}{a} = \frac{\lambda s}{a}$.

Amplitude division using a thin film.
The total phase difference is given by



$$\delta = \frac{2\pi}{\lambda}\left(2n_f d \cos\theta_t\right) + \epsilon = \frac{4\pi n_f d \cos\theta_t}{\lambda} + \epsilon$$

Here, the phase differences by inherent differences and reflections must be considered carefully. Both waves undergo one reflection, but a phase different might occur. To determine the phase changes due to reflections, we must consider the values for n_i, n_f, and n_t. Consider first the case when the film (plate) is in air. The reflection off the front surface is an external reflection (small to larger refractive index) and does involve a phase change of π. The reflection from the back surface, however, is an internal reflection (high to low refractive index) and does not involve a phase change. Therefore, for this case,

$$\delta = \frac{4\pi n_f d \cos\theta_t}{\lambda} \pm \pi.$$

If, however, $n_t > n_f > n_i$, then both reflections would be external and would either cancel or give $2\,\pi$, which does not really matter. Any time both reflections are either internal or external, we would not need to consider the phase change from reflection. However, if one is external and one is internal, there will be a π phase change that must be used the in total phase change equation. Our condition for maxima is

$$\delta = \frac{4\pi n_f d \cos\theta_t}{\lambda} \pm \pi = 2m\pi$$

and

$$d\cos\theta_t = (2m+1)\frac{\lambda}{4n_f} = \frac{(2m+1)}{4}\lambda_f.$$

Minima will occur at

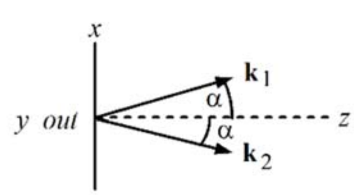
$$\delta = \frac{4\pi n_f d \cos\theta_t}{\lambda} \pm \pi = (2m+1)\pi$$

and

$$d\cos\theta_t = 2m\frac{\lambda_f}{4}.$$

For both internal or both external reflections, the situations are reversed.

Interference using two plane waves.



$$I = I_1 + I_2 + 2\boldsymbol{E}_{o1}\cdot\mathbf{E}_{o2}\cos(2k\,\mathrm{x}\sin\alpha).$$

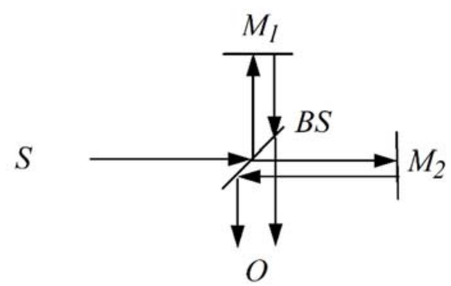
The total phase difference is given by $\delta = 2k\,\mathrm{x}\sin\alpha$, so maxima occur at $\delta = 2m\pi$ and minima occur at $\delta = (2m+1)\pi$. We convert to wavelength and write

$$\frac{2(2\pi)}{\lambda}\sin\alpha\ x_m = 2m\pi \Longrightarrow\ x_m = \frac{m\lambda}{2\ \sin\alpha}.$$

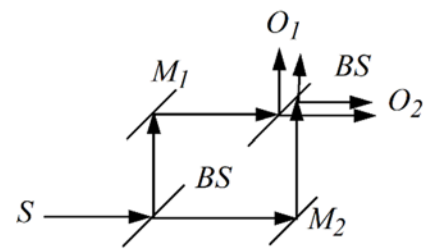
The spacing between fringes is then given by $\Delta x = x_{m+1} - x_m = \frac{\lambda}{2\ \sin\alpha}$. Comment on this result.

Interferometers are extremely useful devices for measuring small differences in quantities. The two most common two-wave interferometers are the Michelson interferometer and the Mach-Zehnder interferometer. Here are figures showing how each one of them functions.

Michelson interferometer



Mach-Zehnder interferometer



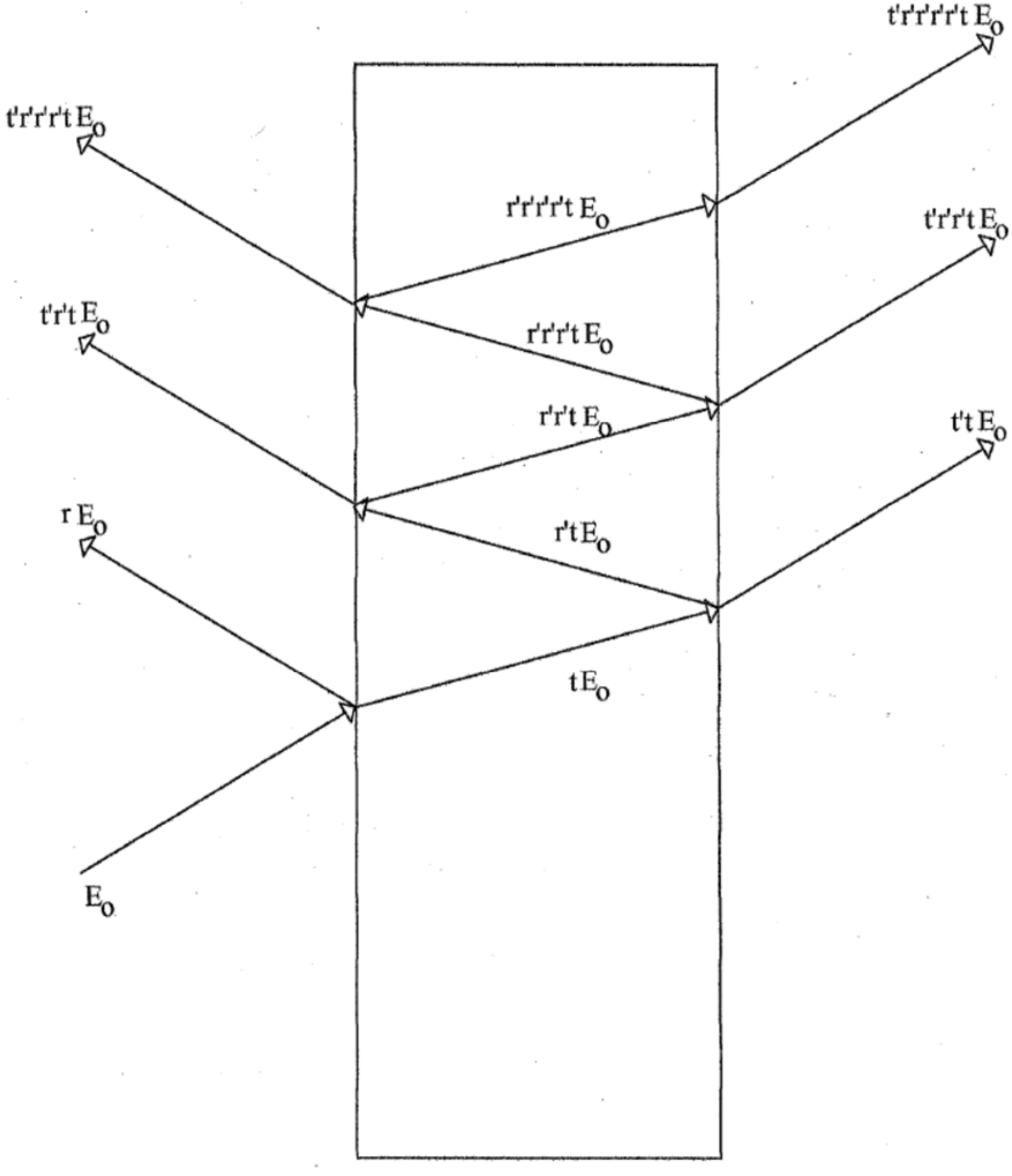
In the Michelson interferometer, the distance between the beam splitter and M_1 is L_1 and the distance between the beam splitter and M_2 is L_2. Therefore, the OPD is just given by $2(L_2\text{-}L_1)$. For the Mach-Zehnder interferometer, the only difference is that each path is traveled only one time, so the OPD is just given by (BS-M_2-BS) – (BS-M_2-BS). For each interferometer, the OPD cannot exceed the coherence length is we expect to observe interference. If the source for each interferometer is a beam of well-collimated light, we will observe interference fringes similar to those in Young's interference; *i.e.*, straight fringes alternating bright and dark. If, however, the source is a point source, we observe circular fringes. If the OPDs are exactly equal, it is possible to observe only one fringe, dark or bright. Give examples of what can be measured. See your textbook on page 418 (5th edition) for a nice way to look at the Michelson interferometer.

Videos 10, 11, and 12.

Multiple wave interference by amplitude division – see next page.

r = external reflection; *r*' = internal reflection; *t* = external transmission; *t*' = internal transmission
Recall internal and external reflections and define internal and external transmissions.

Multiple Wave Interference by a Dielectric Plate



$$E_{or} = E_o r + E_o t'r'te^{-i\delta} + E_o {r'}^3 tt'e^{-2i\delta} + \cdots + E_o tt'r'^{(2N-3)}e^{-i\delta(N-1)}.$$

and

$$E_{or} = E_o\left\{r + r'tt'e^{-i\delta}\left[1 + r'^2e^{-i\delta} + \left(r'^2e^{-i\delta}\right)^2 + \cdots + \left(r'^2e^{-i\delta}\right)^{(N-2)}\right]\right\}$$

The term in the square brackets is a sum that has the form $1 + x + x^2 + \cdots$, which is recognized as $\frac{1}{1-x}$.
Therefore, we obtain

$$E_{or} = E_o\left[r + \frac{r' t t' e^{-i\delta}}{1 - r'^2 e^{-i\delta}}\right].$$

We may use our expressions for the amplitude reflection and transmission coefficients to show that $r = -r'$and $tt' = 1 - r^2$. Making these substitutions gives

$$E_r = E_o\left[r - \frac{r(1-r^2)e^{-i\delta}}{(1 - r^2 e^{-i\delta})}\right]$$

$$= E_o\left[\frac{r\left(1 - r^2 e^{-i\delta}\right) - r(1-r^2)e^{-i\delta}}{(1 - r^2 e^{-i\delta})}\right]$$

$$= E_o\left[\frac{r - r^3 e^{-i\delta} - r e^{-i\delta} + r^3 e^{-i\delta}}{(1 - r^2 e^{-i\delta})}\right]$$

$$= E_o\left[\frac{r\left(1 - e^{-i\delta}\right)}{(1 - r^2 e^{-i\delta})}\right].$$

Now, we need to get the irradiance which we know is given by

$$I_r = \frac{1}{2} Re\ (E_r E_r^*) = I_i \frac{2r^2(1 - \cos\delta}{(1 + r^4 - 2r^2\cos\delta)} \text{ with } I_i = \frac{E_o^2}{2}.$$

If we ignore absorption,

$$I_t = I_i \frac{(1-r^2)^2}{(1 + r^4 - 2r^2\cos\delta)}$$

Now use $\cos\delta = 1 - 2\sin^2\frac{\delta}{2}$ to obtain

$$I_r = I_i \frac{\left[\frac{2r}{(1-r^2)}\right]^2 \sin^2\frac{\delta}{2}}{1 + \left[\frac{2r}{(1-r^2)}\right]^2 \sin^2\frac{\delta}{2}} \text{ and } I_t = I_i \frac{1}{1 + \left[\frac{2r}{(1-r^2)}\right]^2 \sin^2\frac{\delta}{2}}.$$

As we would expect $I_r + I_t = I_i$ if no absorption is present. Let's examine the maxima and minima of these equations.

$$(I_t)_{max} = I_i \text{ when } \delta = 2m\pi \text{ and } (I_r)_{min} = 0$$

If, however, $\delta = (2m+1)\pi$ so that $\frac{\delta}{2} = (2m+1)\frac{\pi}{2}$

$$(I_t)_{min} = I_i \frac{(1-r^2)^2}{(1+r^2)^2} \text{ and } (I_r)_{max} = I_i \frac{4r^2}{(1+r^2)^2}$$

Define $F = \left(\frac{2r}{1-r^2}\right)^2$ = coefficient of finesse and write

$$\frac{I_r}{I_i} = \frac{F\sin^2\frac{\delta}{2}}{1+F\sin^2\frac{\delta}{2}};\ \ \frac{I_t}{I_i} = \frac{1}{1+F\sin^2\frac{\delta}{2}};\ \ \frac{1}{1+F\sin^2\frac{\delta}{2}} = \text{Airy function}$$

Sometimes, you will write the finesse as $\mathscr{F} = \frac{\pi}{2}\sqrt{F}$. But much of this is just renaming things to agree with historical naming of terms.

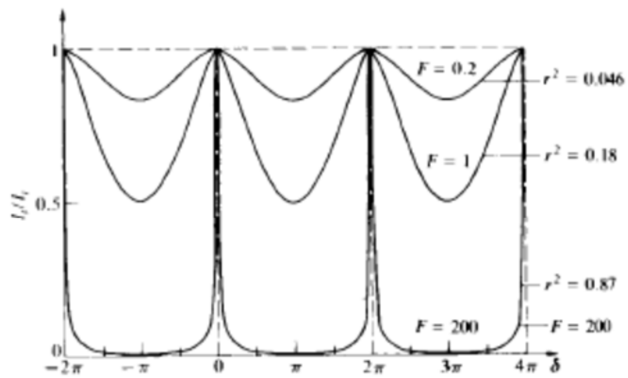
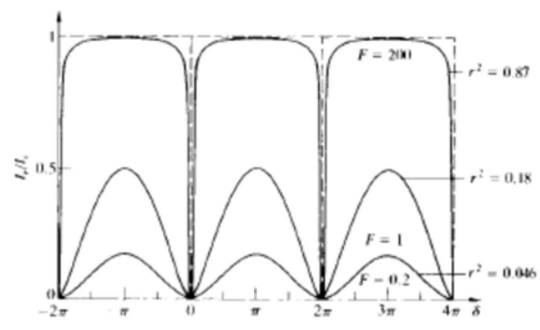


Figure 9.41 Airy function.



igure 9.42 One minus the Airy function.

NEXT TIME: We will try to understand better the reasons for the dramatic change in width of the minima and maxima as the reflection coefficient becomes larger and larger. We will also begin multiple wave interference by wavefront division.\

REMINDER: Examination 2 is on Wednesday, March 27 and will cover waves and polarization. I will post the next homework set soon because we are somewhat ahead in lecture of our homework.