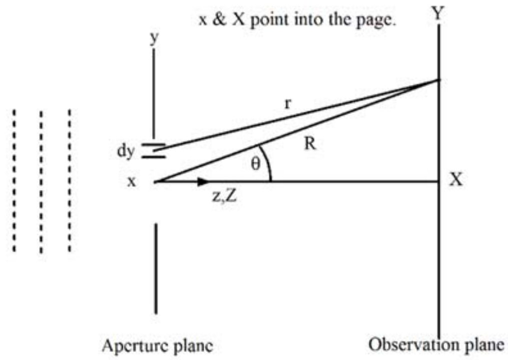
LAST TIME: Fraunhofer diffraction and multiple-aperture interference

REMINDER:



$$E(Y) = \int \mathcal{E}_L\, \mathcal{A}(y) \frac{e^{-ikr}}{r} dy,$$

where $\mathcal{E}_L$ is the source strength per unit length and $\mathcal{A}(y)$ is the aperture function. With suitable approximations we found

$$E(Y) = \frac{\mathcal{E}_L e^{-ikR}}{R} \int_{-\infty}^{\infty} \mathcal{A}(y)\, e^{ikyY/R} dy.$$

$$kR\left(\frac{1}{2}\right)\left(\frac{y}{R}\right)^2 \ll 1 \implies \frac{2\pi}{\lambda} R \left(\frac{1}{2}\right)\left(\frac{b}{R}\right)^2 \ll 1 \implies \frac{\lambda R}{b^2} \gg 1 \implies \lambda \gg \frac{b^2}{R}.$$

Single slit diffraction example

$$I(\theta) = \left(\frac{1}{2}\right) Re(E^* E) = \left(\frac{1}{2}\right)\left(\frac{\mathcal{E}_L b}{R}\right)^2 \left(\frac{\sin\beta}{\beta}\right)^2 = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$$

Minima in the diffraction pattern occur whenever $\beta = m\pi, m \neq 0$. Therefore, $\frac{kb\sin\theta}{2} = m\pi$ so $b\sin\theta = m\,\lambda, m \neq 0$ is the condition for each minimum. To get the maxima, we must calculate the dreivative, set it equal to zero, and solve the equation. This gives the result that

$$\frac{dI}{d\beta} = 0 \implies \tan\beta = \beta.$$

This is a transcendental equation and is solved numerically to obtain values given by

$$\beta = \pm 1.43\pi, \pm 2.45\pi, \pm 3.47\pi$$

so that

$$\beta \cong \left(m + \frac{1}{2}\right)\pi \text{ with } m = 1,2,\cdots$$

For two-wave diffraction-interference we found that

$$I(\theta) = 4I_o \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha \text{ with } \alpha = \frac{ka\sin\theta}{2}.$$

Therefore,

$$I(\theta) = 0 \text{ for } \beta = \pm m\pi \;\; (m \neq 0) \text{ or } \alpha = \pm\left(m + \frac{1}{2}\right)\pi.$$

For multiple-wave interference-diffraction

$$E = \mathcal{E}_L b \left(\frac{\sin\beta}{\beta}\right)\frac{e^{-ikR}}{R}\frac{e^{i\alpha N}}{e^{i\alpha}}\left[\frac{e^{i\alpha N} - e^{-i\alpha N}}{e^{i\alpha} - e^{-i\alpha}}\right]$$

Now get irradiance given by

$$I = \frac{1}{2}Re(E^*E) = I_o\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2,$$

Principal maxima occur when $\alpha = 0, \pm\pi, \pm 2\pi, etc.$ The diffraction zeros will always be determined by the slit width regardless of how many slits there are. Minima in the irradiance occur when $\left(\frac{\sin N\alpha}{\sin\alpha}\right) = 0$ or at values of $\alpha = \pm\frac{\pi}{N}, \pm\frac{2\pi}{N}, \frac{3\pi}{N}, \cdots, \pm\frac{(N-1)\pi}{N}, \pm\frac{(N+1)\pi}{N}$.

Between successive principal maxima, N-1 minima occur, and between each pair of principal maxima, N-2 subsidiary maxima occur.

Two-dimensional diffraction

Our diffraction integral now becomes two-dimensional so, proceeding as before, we see the figure to dexcribe this effect as shown to the right. Now

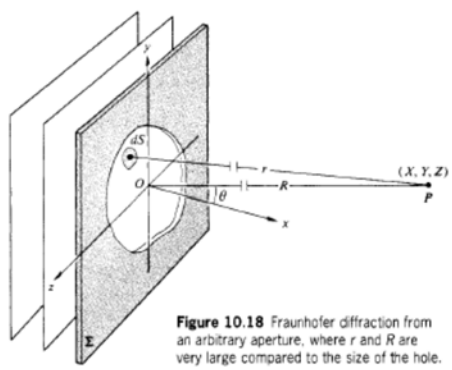


Figure 10.18 Fraunhofer diffraction from an arbitrary aperture, where *r* and *R* are very large compared to the size of the hole.

$$r = [X^2 + (Y-y)^2 + (Z-z)^2],$$

so approximately,

$$r = R[1 - (Yy + Zz)/R^2]$$

and

$$E(Y,Z) = \frac{\mathcal{E}_A e^{-ikR}}{R}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{A}(y,z)\, e^{ik(yY+zZ)/R}dydz,$$

Where I have again let the limits of the integration be set by our aperture function. If we assume we have a rectangular aperture with dimensions *b* by *a*, then our aperture function will be given by

$$\mathcal{A}(y,z) = 1 \text{ for } -\frac{b}{2} \le y \le \frac{b}{2}$$
$$\text{and } -\frac{a}{2} \le z \le \frac{a}{2}$$
$$= 0 \text{ elsewhere.}$$

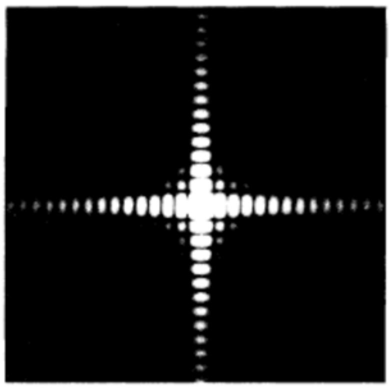
Using this aperture function, our integral becomes

$$E(Y,Z) = \frac{\mathcal{E}_A e^{-ikR}}{R} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{ik(yY+zZ)/R} dydz\,,$$

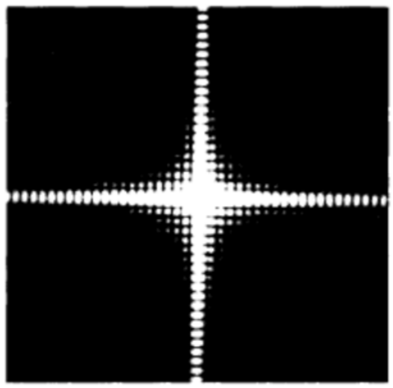
Make the substitution $\beta' = \frac{kby}{2R}$ and $\alpha' = \frac{kaz}{2R}$ to obtain

$$I(Y,Z) = I(0)\left(\frac{\sin\alpha'}{\alpha'}\right)^2 \left(\frac{\sin\beta'}{\beta'}\right)^2.$$

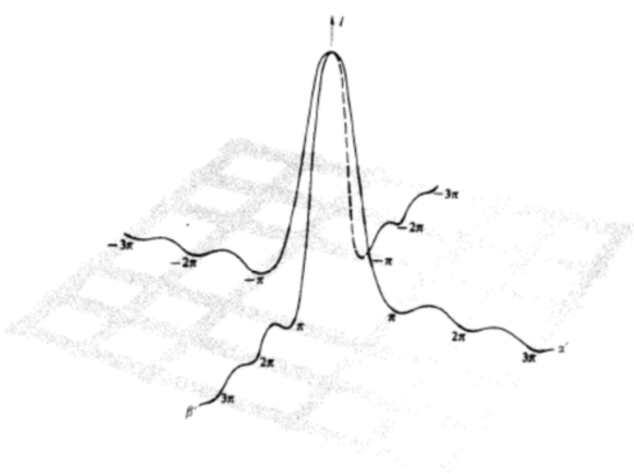
Some figures from Hecht show the behavior.



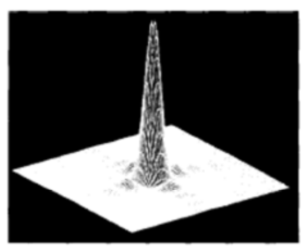
(a)



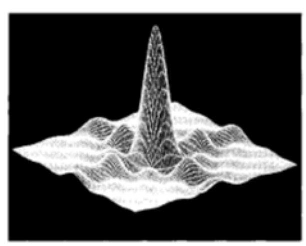
(b)



(a)



(b)



(c)

Figure 10.20 (a) The irradiance distribution for a square aperture. (b) The irradiance produced by Fraunhofer diffraction at a square aperture. (c) The electric-field distribution produced by Fraunhofer diffraction via a square aperture. (Photos courtesy R. G. Wilson, Illinois Wesleyan University.)

Circular apertures are much more interesting and useful because they are the shape of most lenses. The geometry is shown on the following page.

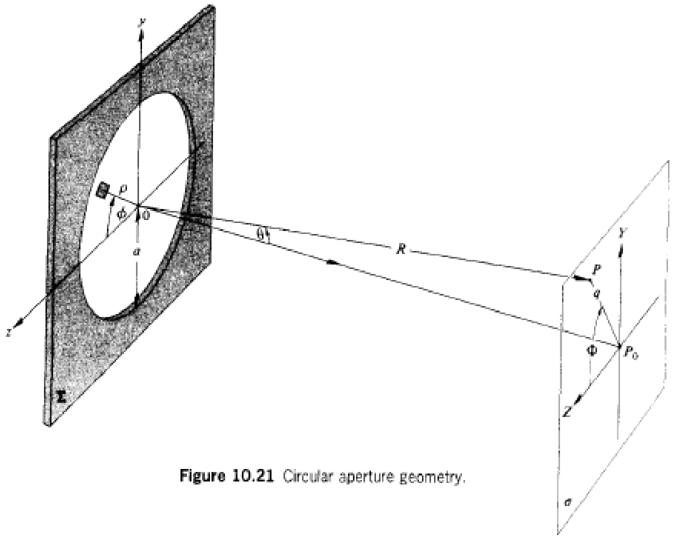


Figure 10.21 Circular aperture geometry.

From before, we obtained

$$E(Y,Z) = \frac{\mathcal{E}_A e^{-ikR}}{R}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}(1)e^{ik(yY+zZ)/R}dydz\,.$$

It makes no sense to try to deal with a circular aperture using Cartesian coordinates, so we change to polar coordinates using the following transformations.

$$z = \rho\cos\phi;\;\; y = \rho\sin\phi\,;\;\; Z = q\cos\Phi\,;\;\; Y = q\sin\Phi$$

The area element in polar coordinates is given by $dS = \rho d\rho d\phi$. Therefore, our integral in polar coordinates is given by

$$E(q,\phi) = \frac{\mathcal{E}_A e^{-ikR}}{R}\int_0^a\int_0^{2\pi}\mathcal{A}(\rho,\phi)e^{i\left(\frac{k\rho q}{R}\right)\cos(\phi-\Phi)}\rho d\rho d\phi.$$

Most of the time, our optical systems are not dependent on ϕ, so $\mathcal{A}(\rho,\phi) = \mathcal{A}(\rho)$. The we may choose $\Phi = 0$ without loss of generality. $\mathcal{A}(\rho) = 1$ for $0 \le \rho \le a$ and $= 0$ elsewhere. Now we need to delve into Bessel functions to get a clue about how to proceed. Comments. We note that

$$J_m(u) = \frac{i^{-m}}{2\pi}\int_0^{2\pi}e^{i(mv+u\cos v)}\,dv.$$

Therefore

$$E(q,\phi) = \frac{\mathcal{E}_A e^{-ikR}}{R}(2\pi)\int_0^a J_o\left(\frac{k\rho q}{R}\right)\rho d\rho$$

Use an identity

$$\int_0^a u' J_o(u')\,du' = uJ_1(u)$$

to obtain

$$\int_0^a J_o\left(\frac{k\rho q}{R}\right)\rho d\rho = \left(\frac{R}{kq}\right)^2 \int_0^{\frac{kaq}{R}} J_o\,(w)wdw$$

Finally,

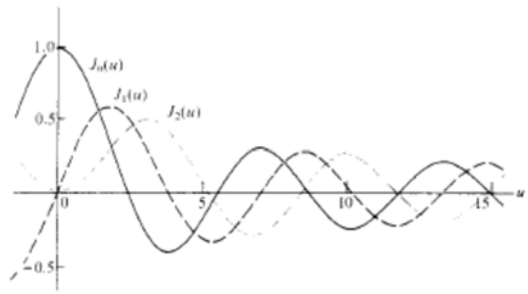
$$E(q,\phi) = \frac{\mathcal{E}_A e^{-ikR}}{R}(2\pi)a^2\left(\frac{R}{kaq}\right)J_1\left(\frac{kaq}{R}\right)$$

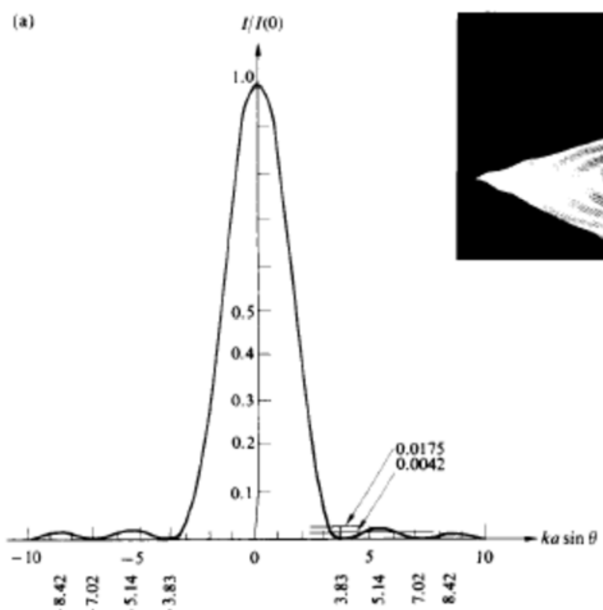
and

$$I(q) = \left(\frac{2\mathcal{E}_A{}^2A^2}{R^2}\right)\left[\frac{J_1(kaq/R)}{kaq/R}\right]^2.$$

$$I(\theta) = I(0)\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]^2$$

where $\sin\theta = \frac{q}{R}$ and $I(0) = \frac{2\mathcal{E}_A{}^2A^2}{R^2}$. To find the zeroes of I, we must find the value of ka sin θ that makes the Bessel function zero. The values of the function for different arguments are in tables, so we look to see where the Bessel function is zero and note that argument. From the table or graph, we wee that the first zero occurs at 3.83, so $\frac{kaq}{R} = \frac{2\pi aq}{\lambda R} = 3.83$ and $q_1 = \frac{\lambda R}{2a}(1.22)$. The folowing graphs from your textbook show some of the important features of diffraction by a circular aperture. The very high central peak is know as the Airy disk, named after the person who first derived the expression.





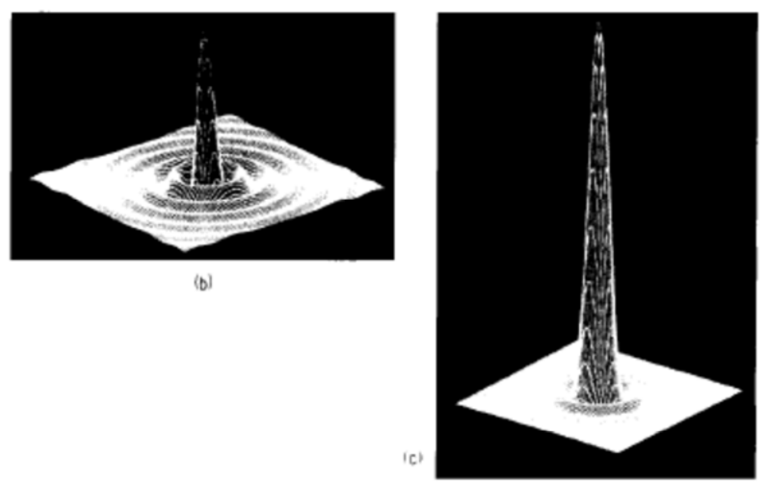


Figure 10.23 (a) The Airy pattern. (b) Electric field created by Fraunhofer diffraction at a circular aperture. (c) Irradiance resulting from Fraunhofer diffraction at a circular aperture. (Photos courtesy R. G. Wilson, Illinois Wesleyan University.)

On of the most important applications of this result occurs when we try to resolve to objects very close together. If a lens is placed in the aperture, then the value for R becomes the focal length f

of the lens, and 2*a* is the diameter of the lens. This means that the radius of the first bright spot is given by

$$q_1 = 1.22\frac{f\lambda}{D} = \frac{(1.22)(127\text{ mm})633 \times 10^{-6}\text{mm}}{25.4\text{ mm}} = 4\text{ micrometers}$$

This means that if you filled the lens with the laser beam, the focused spot size would be about 4 micrometers in radius. Unfortunately, however, the unexpanded laser beam does not fill the entire lens, so if the laser beam is onlyl about 1 mm in diameter, then the radius of the central spot is about 100 micrometers. This example shows the advantage of using large lenses and completely filling the aperture with the beam. It also illustrates why the blue-ray dvds are so useful. The shorter the wavelength, the smaller the radius of the central spot size, all other things being equal.

Comments on eyes and animal eyes and resolution.

NEXT TIME: Diffraction from barriers and Fresnel diffraction with circular apertures.