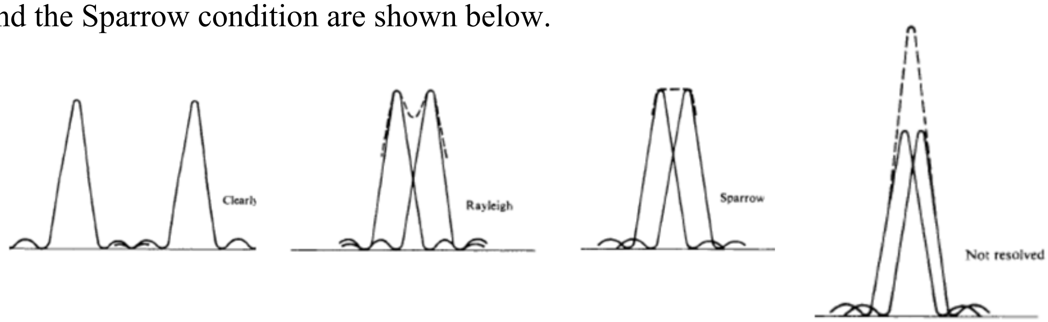
LAST TIME: Resolution, Fresnel diffraction with plane waves, Fresnel diffraction with spherical waves, barriers (return to barriers later)

REMINDER:

Resolution

The Rayleigh criterion for resolving two objects is to state that the minimum of one of the diffraction patterns just overlaps the maximum of the other object. The figures for this condition and the Sparrow condition are shown below.



We found that $q_1 = \frac{\lambda R}{2a}(1.22)$ and with a lens, $q_1 = 1.22\frac{f\lambda}{D}$.

Therefore, we may also use an angular measure as the resolution limit. Then

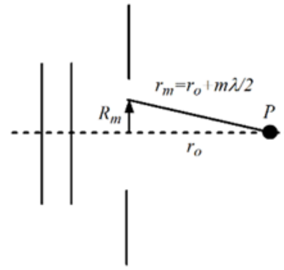
$$\Delta\theta = \frac{q_1}{f} = 1.22\frac{\lambda}{D},$$

so for a wavelength of 550 nm and a lens or mirror diameter D = 3.5 m, which is the diameter of the main adaptive optics telescope at the Starfire Optical Range I mentioned earlier, its diffraction limit of Rayleigh resolution is given by

$$\Delta\theta = \frac{q_1}{f} = 1.22\frac{\lambda}{D} = 1.22\left(\frac{550 \times 10^{-9}}{3.5}\right) = 1.92 \times 10^{-7}\text{ rad.}$$

There are 57.2 degrees/rad, and 3600 arc seconds/degree, so this resolution is about 0.04 arc seconds. This telescope has separated binary stars with a separation of 0.1 arc seconds. For an object 100 miles away or about 160 km =1.6 x 10^5 m away, the telescope could resolve points on the object about 0.03 m or 3 cm apart. When optical devices are able to have this type of resolution, we say that they have diffraction limited resolution.

Fresnel diffraction with plane waves incident.



$$r_o^2 + R_m^2 = r_m^2 = \left(r_o + \frac{m\lambda}{2}\right)^2 = r_o^2 + m r_o \lambda + \frac{m^2\lambda^2}{4}$$

so

$$R_m \cong \sqrt{m r_o \lambda}\; with\; \frac{m^2\lambda^2}{4} \ll 1.$$

$$A = A_{m+1} - A_m = \pi(R_{m+1}^2 - R_m^2) = \pi(m+1)r_o\lambda - \pi m r_o \lambda = \pi r_o \lambda.$$

$$= \frac{|E_1|}{2} + \left(\frac{|E_1|}{2} - |E_2| + \frac{|E_3|}{2}\right) + \cdots + \left(\frac{|E_{m-2}|}{2} - |E_{m-1}| + \frac{|E_m|}{2}\right) \pm \frac{|E_m|}{2}.$$

Each of the terms in parenthesis gives approximately zero, so

$$E_u = \frac{|E_1|}{2} \pm \frac{|E_m|}{2}.$$

Once m becomes very large, the obliquity factor begins to reduce its value, so we may neglect the last term in the sum above. Therefore,

$$E_u = \frac{E_1}{2} \Longrightarrow I_1 = 4I_u$$

$$R_1 \cong \sqrt{r_o\lambda} = D,$$

where D is the diameter of the aperture. This means that $D^2 = r_o\lambda$, but recall that our condition for Fraunhofer diffraction was given by

$$\lambda \gg \frac{b^2}{R}.$$

Fresnel diffraction with spherical wave incident

$$\rho_m^2 = \rho_o^2 + R_m^2 \;\; and \; r_m^2 = r_o^2 + R_m^2$$

$$\frac{R_m^2}{\rho_o} + \frac{R_m^2}{r_o} = \; m\lambda \;\; or \;\; \frac{1}{\rho_o} + \frac{1}{r_o} = \frac{m\lambda}{R_m^2}.$$

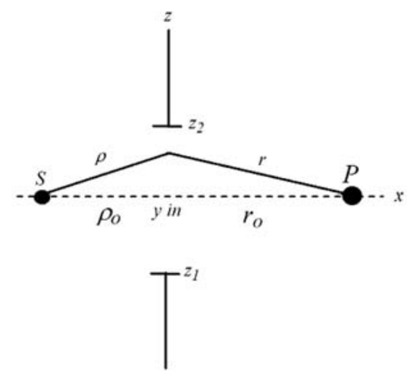
The physics here is the same, and calculating the area of each zone proceeds in the same way. It turns out that the area of eahc zone is given by

$$A = \left(\frac{\rho_o}{\rho_o + r_o}\right)\pi r_o \lambda.$$

$$Nzones = \frac{\pi R^2}{\left(\frac{\rho_o}{\rho_o + r_o}\right)\pi r_o \lambda} = \frac{(\rho_o + r_o)R^2}{\rho_o r_o \lambda}.$$

Fresnel diffraction using rectangular barriers.

Comment on problem using Fresnel zone approach. Go back to the original diffraction integral and keep additional term in expansion. Our figure remains essentially the same, but the aperture is now rectangular instead of circular. For this case, however, it turns out to be easier to calculate the diffraction effects on the axis and then allow the aperture to move to obtain the off-axis diffraction irradiance. It is not necessary to do the problem this way, but it is easier to deal with byt doing so. Our diffraction integral becomes



$$E_P = \int_{y_1}^{y_2} \int_{z_1}^{z_2} \mathcal{E}_o \frac{e^{i[k(\rho + r)]}}{\rho r \lambda} dS,$$

Where the harmonic time dependence has been ommitted and the oblilquity factor has been set equal to one, a reasonable assumption in most cases. Now we note that

$$\rho = ({\rho_o}^2 + y^2 + z^2)^{1/2} \text{ and } r = ({r_o}^2 + y^2 + z^2)^{1/2}.$$

Approximately,

$$\rho \cong \rho_o \left[1 + \frac{(y^2 + z^2)}{2\rho_o}\right] \text{ and } r \cong r_o \left[1 + \frac{(y^2 + z^2)}{2r_o}\right]$$

so that

$$\rho + r = \rho_o + r_o + (y^2 + z^2)\left(\frac{1}{2\rho_o} + \frac{1}{2r_o}\right) = \rho_o + r_o + (y^2 + z^2)\left(\frac{\rho_o + r_o}{2\rho_o r_o}\right).$$

We substitute all this back into our diffraction integral for calculating the on-axis diffraction field to obtain

$$E_P = \frac{\mathcal{E}_o}{\rho_o r_o \lambda} \int_{y_1}^{y_2} \int_{z_1}^{z_2} e^{i\left[k\left(\rho_o + r_o + (y^2+z^2)\left(\frac{\rho_o + r_o}{2\rho_o r_o}\right)\right)\right]} dzdy.$$

Simplifying gives

$$E_P = \frac{\mathcal{E}_o}{\rho_o r_o \lambda} e^{ik(\rho_o + r_o)} \int_{y_1}^{y_2} \int_{z_1}^{z_2} e^{i\left[k\left((y^2+z^2)\left(\frac{\rho_o + r_o}{2\rho_o r_o}\right)\right)\right]} dzdy.$$

Now we make some substitutions that make this integral look a bit more standard. We let

$$u = y\left[\frac{2(\rho_o + r_o)}{\rho_o r_o \lambda}\right]^{1/2} \text{ and } v = z\left[\frac{2(\rho_o + r_o)}{\rho_o r_o \lambda}\right]^{1/2}$$

to obtain

$$E_P = \frac{\mathcal{E}_o}{2(\rho_o + r_o)} e^{ik(\rho_o + r_o)} \int_{u_1}^{u_2} e^{i\pi u^2/2}\, du \int_{v_1}^{v_2} e^{i\pi v^2/2}\, dv.$$

These integrals over u and v are known as the Fresnel integrals. Now we define the following functions as

$$\mathcal{C}(w) = \int_0^w \cos\left(\frac{\pi w'^2}{2}\right) dw' \text{ and } \mathcal{S}(w) = \int_0^w \sin\left(\frac{\pi w'^2}{2}\right) dw'$$

so that

$$\int_0^w e^{i\pi w'^2/2}\, dw' = \mathcal{C}(w) + i\,\mathcal{S}(w)$$

Now we may expand all this to obtain

$$E_P = \frac{E_u}{2}\left[\mathcal{C}(u) + i\,\mathcal{S}(u)\right]_{u_1}^{u_2}\left[\mathcal{C}(v) + i\,\mathcal{S}(v)\right]_{v_1}^{v_2}.$$

Carrying out the algebra

$$E_P = \frac{E_u}{2}\{[\mathcal{C}(u_2) + i\,\mathcal{S}(u_2) - \mathcal{C}(u_1) - i\,\mathcal{S}(u_1)] \times [\mathcal{C}(v_2) + i\,\mathcal{S}(v_2) - \mathcal{C}(v_1) - i\,\mathcal{S}(v_1)]\}$$

Therefore,

$$I_P = \frac{I_o}{4}\{[\mathcal{C}(u_2) - \mathcal{C}(u_1)]^2 + [\mathcal{S}(u_2) - \mathcal{S}(u_1)]^2\} \times \{[\mathcal{C}(v_2) - \mathcal{C}(v_1)]^2 + [\mathcal{S}(v_2) - \mathcal{S}(v_1)]^2\}.$$

The $\mathcal{C}$ and $\mathcal{S}$ are the Fresnel cosine and sine integrals, and their values are tabulated. Many mathematical assistant programs will calculate them very quickly. u_1 and u_2 are rescaled values for y_2 and y_1, and v_1 and v_2 are rescaled values for z_1 and z_2. Let's do an example just to get our feet wet before we start on more complicated calculations that are off – axis.

Example

Consider a square aperture whose side length is 1.00 mm with both ρ_o and r_o equal to 0.5 m. First, we should check to see if we need to use Fresnel diffraction or not. Notice that

$$\lambda(\rho_o \text{ } or \text{ } r_o) = 500 \times 10^{-9} \times 0.5 = 2.5 \times 10^{-7}\text{m}^2,$$

but

$$b^2 = (10^{-3})^2 = 10^{-6}\text{m}^2 \text{ so } b^2 > \lambda r_o \implies \text{Fraunhofer condition not met}$$

Therefore,

$$y_1 = -0.5 \text{ mm}, \qquad y_2 = +0.5 \text{ mm}, \qquad z_1 = -0.5 \text{ mm}, \qquad z_2 = +0.5 \text{ mm}$$

Calculate

$$u_1 = y_1\left[\frac{2(\rho_o + r_o)}{\rho_o r_o \lambda}\right]^{\frac{1}{2}} = -0.5 \times 10^{-3}\left[\frac{2(0.5 + 0.5)}{(0.5)(0.5)(500 \times 10^{-9})}\right] = -2.0$$

By symmetry

$$u_2 = +2.0 \quad v_1 = -2.0 \quad v_2 = +2.0$$

We may use the fact that $\mathcal{C}(w) = -\mathcal{C}(-w)\ and\ \mathcal{S}(w) = -\mathcal{S}(-w)$ to write the irradiance at P as

$$I_P = \frac{I_o}{4}\{[2\mathcal{C}(u_2)]^2 + [2\mathcal{S}(u_2)]^2\}^2 = \frac{I_o}{4}\{[2 \times 0.4882]^2 + [2 \times 0.3434]^2\}^2$$

$$= \frac{I_o}{4}\{0.9534 + 0.4717\}^2 = 0.51 I_o.$$

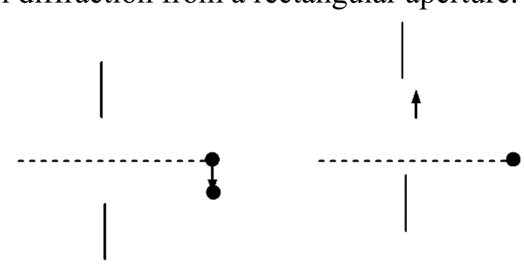
Check to see whaat we get with no aperture in place. With no aperture, our aperture extends from –infinity to + infinity in each direction, and with $\mathcal{C}(w) = \ \mathcal{S}(w) = 0.5$,

$$I_P = \frac{I_o}{4}\{[2(1/2)]^2 + [2(1/2)]^2\}^2 = I_o.$$

Suppose we want to use plane waves instead of a point source. All we need to do is let $\rho_o \to \infty$, and to obtain the corresponding values to u and v given by

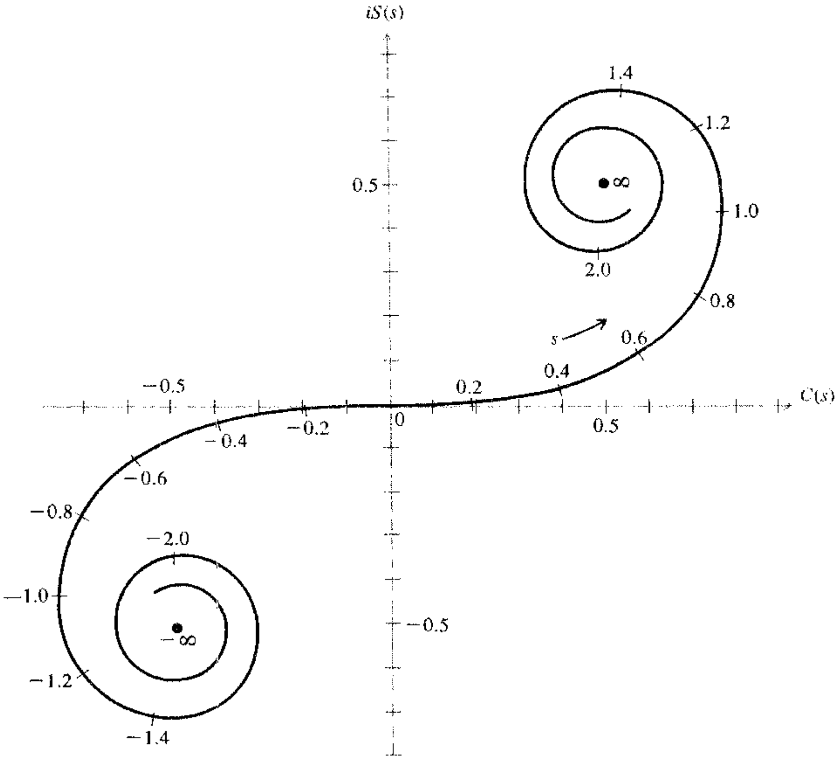
$$u = y\left[\frac{2}{r_o\lambda}\right]^{1/2} \text{and } v = z\left[\frac{2}{r_o\lambda}\right]^{1/2}.$$

It is not particularly useful to calculate only the irradiance on the axis of symmetry, so we need to find a way to calculate patterns off the symmetry axis. There are two ways of doing this. One method is to build into the diffraction integral an offset that allows us to do this directly, but it turns out that a better and more instructive method is to recognize that moving off axis is equivalent to relocating the aperture in the opposite direction to the way we moved our observation point. Here is a graph showing how this idea works. Then we will look at a graphical method of doing these calculations using a plot of $\mathcal{S}(w)$ versus $\mathcal{C}(w)$. This graph is called the Cornu spiral and is quite useful in visualizing what is going on with Fresnel diffraction from a rectangular aperture. The left part of the figure shows that new relative location of the observation point with respect to the lower boundary of the aperture. The right side shows the observation point remainging fixed while moving the aperture up by the same distance the observation point was moved downward. The relative position of the observation point and the boundaries of the aperture are the same in each case.



Using this approach allows us to visualize the process using the Cornu spiral. By the way, the Cornu spiral is just the vibration curve for the rectangular geometry as the spiral I showed for the circular aperture. The following page shows each of the vibration curves, the Cornu sprial and the more traditional spiral.

Cornu spiral



Circular apeture vibration curve